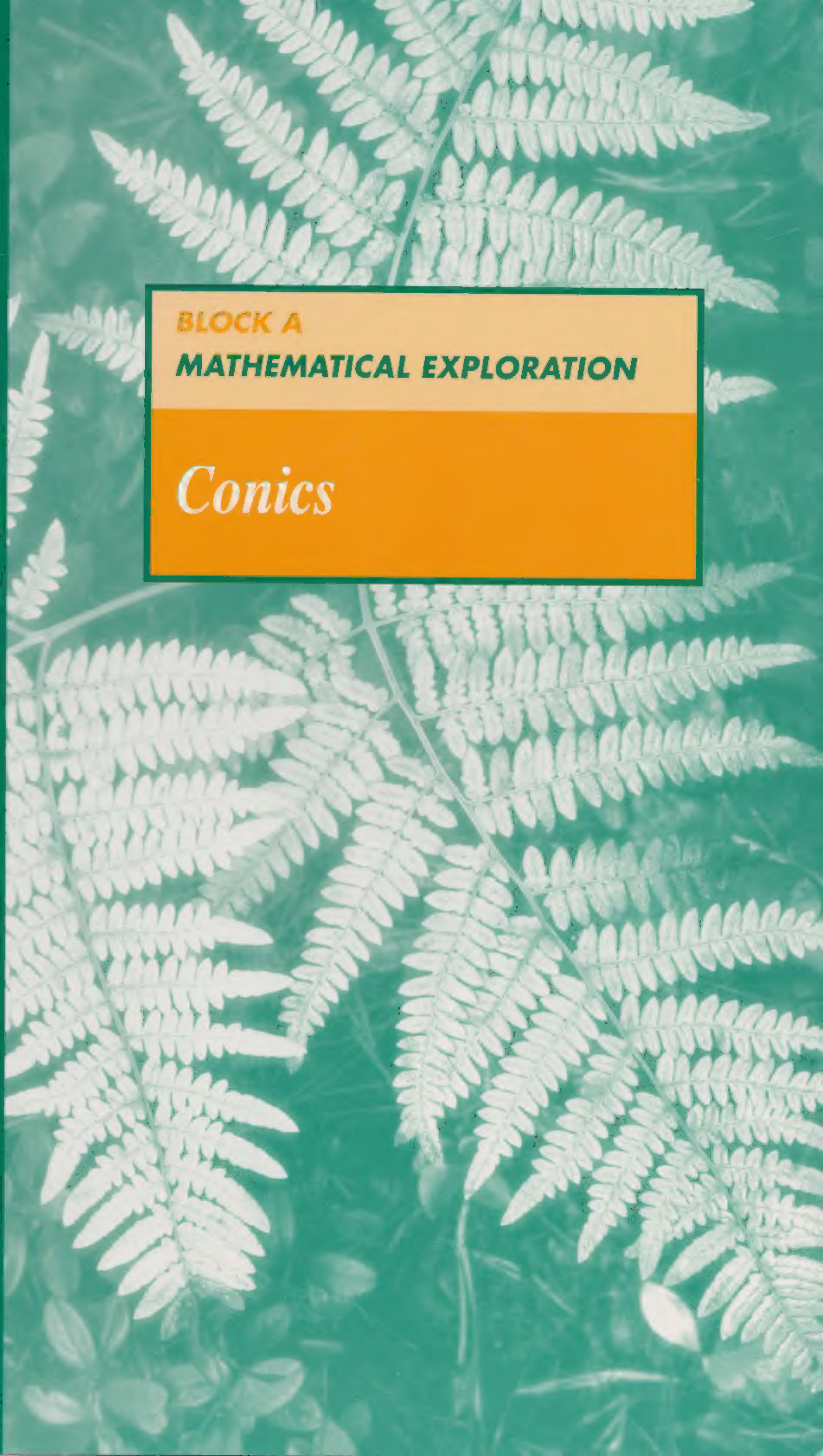
MS221 Chapter A2

The Open University

A second level interdisciplinary course

Exploring Mathematics



BLOCK A

MATHEMATICAL EXPLORATION

Conics

CHAPTER

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Conics

Prepared by the course team

About this course

This course, MS221 *Exploring Mathematics*, and the courses MU120 *Open Mathematics* and MST121 *Using Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MS221 uses the software program Mathcad (MathSoft, Inc.) to investigate mathematical concepts and as a tool in problem solving. This software is provided as part of the course.

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Contents

Study guide

There are six sections to this chapter. They are intended to be studied consecutively, in four study sessions. Section 1 requires the use of a video player, Section 4 requires the use of an audio cassette player, and Section 6 requires the use of the computer together with Computer Book A.

The pattern of study for each session might be as follows.

Study session 1: Section 1 and Section 2.

Study session 2: Section 3.

Study session 3: Section 4.

Study session 4: Section 5 and Section 6.

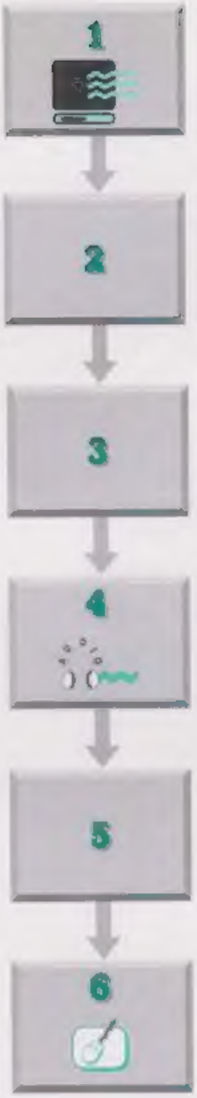
Study session 3 requires about 1–2 hours; the other sessions require about 3 hours each.

Before studying this chapter, you should be familiar with the following topics:

- the formula for the distance between two points in a plane;
- the equation of a line and the equation of a circle;
- completing the square in a quadratic expression;
- the idea of parametrisation, and the parametric equations of a circle;
- the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$;
- the trigonometric ratio
  $$\sec\theta = \frac{1}{\cos\theta}.$$

In this chapter the symbol $\simeq$, which is read as 'is approximately equal to', is used. The approximations are correct to the number of significant figures given.

> The optional Video Band A(ii) *Algebra workout – Square roots* could be viewed at any stage during your study of this chapter.



Introduction

This chapter is about three types of curve – the *ellipse*, the *hyperbola* and the *parabola*.

Pronounce each of these names with emphasis on their second syllable.

A *circle* is a special type of ellipse.

Figure 0.1 Conic sections

These types of curve have been known for over two thousand years, since the time of the Ancient Greeks. They are called *conic sections* – or *conics*, for short – since they can all be obtained by slicing a cone in various ways.

Conic sections were probably introduced by Menaechmus (fourth century BC), and were studied in depth by Apollonius of Perga (third century BC).

The conic sections are of interest because of their appearances in a wide variety of situations. For example, ellipses occur as the orbits of planets and comets, hyperbolas as the paths of subatomic particles, and parabolas as the curves traced by objects moving under the influence of the Earth's gravity. Conics are also of interest algebraically. Just as linear equations (such as the equation $y = 3x - 4$) represent lines, so quadratic equations (such as the equation $4x^2 + y^2 - 2y = 1$) generally represent conics.

We start, in Section 1, by exploring what happens when we slice various three-dimensional objects, such as spheres and cylinders. When we slice a cone we obtain the conic sections. Section 1 concludes with a more detailed description of the above occurrences of conics.

In Section 2, the main geometric and algebraic properties of conics in *standard position* are described. In spite of their differing appearances, you will see that ellipses, hyperbolas and parabolas have many features in common.

In Section 3, we find that all three types of conic have the so-called *focus–directrix property*. This provides an alternative geometric definition of conics and also leads to other elegant properties of conics, which have practical applications.

In Section 4, we consider the general quadratic equation, involving terms in x^2, xy and y^2, as well as linear terms. We explore ways of rearranging certain quadratic equations in order to determine which types of conic they represent.

In Section 5, it is shown how each type of conic can be represented parametrically, and finally, in Section 6, such parametric representations are used to draw conics with the computer.

1 What is a conic?

To study Subsection 1.1, you will need a video player and the Video Tape.

In this section, we first investigate what happens when we slice various objects, such as spheres, cylinders and cones, by a plane. This leads us, in particular, to the conic sections – ellipses, hyperbolas and parabolas. Then some occurrences of the conic sections in science are described.

1.1 Cross-sections

We can often obtain information about a three-dimensional object by considering slices, or *cross-sections*, of it. For some objects, such as a sphere, it is easy to imagine what the cross-sections look like.

Example 1.1 Slicing a sphere

A **cross-section** of an object is the intersection of the object and a plane.

Describe the possible types of cross-sections that can be obtained when we slice a hollow sphere by a plane.

Solution

If we slice a hollow sphere through its centre, then the cross-section is a circle whose diameter is the same as that of the sphere; see Figure 1.1(a).

Note that, since the sphere is hollow, each cross-section is a circle, rather than a disc.

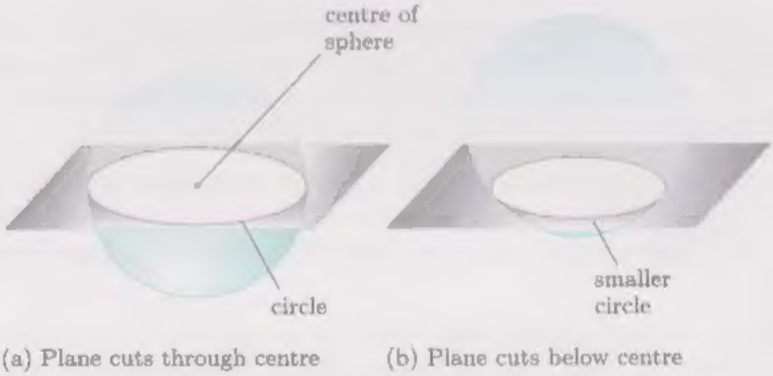


Figure 1.1 Circular cross-sections

If we now lower the plane, so that it no longer passes through the centre of the sphere, then we obtain a smaller circular cross-section; see Figure 1.1(b).

As we continue to lower the plane, the circular cross-section becomes smaller. Eventually the plane no longer passes through the sphere but just touches it; see Figure 1.2(a). In this case, the cross-section is no longer a circle, but is a single point.

(a) Plane touches sphere (b) Plane misses sphere

Figure 1.2 Other cross-sections

In case (a), the plane is called a **tangent plane** to the sphere.

Finally, if we lower the plane still further, then it loses contact with the sphere and the cross-section has no points in it; see Figure 1.2(b). We express this by saying that the cross-section is the *empty set*.

To summarise, the possible cross-sections are: circles whose diameter does not exceed that of the sphere, single points, and the empty set.

If the slicing plane is *not* horizontal, then similar types of cross-sections are obtained.

Now try the following activity, but do not spend more than a few minutes on it.

Activity 1.1 Slicing a cylinder

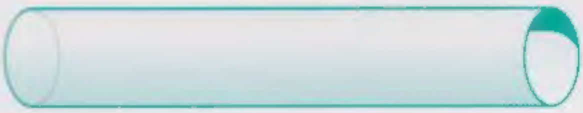


Figure 1.3 Cylinder

Describe the possible types of cross-section that can be obtained when we slice an infinitely long horizontal hollow cylinder (Figure 1.3) by a plane. By the rotational symmetry of the cylinder, you need only consider planes that are

(a) vertical and perpendicular to the axis of the cylinder;

(b) tilted from this vertical position about a horizontal axis which is perpendicular to the axis of the cylinder;

(c) horizontal.

A solution is given on page 49.

In Example 1.1 and Activity 1.1, some of the cross-sections seem more interesting than the others. For the sphere, the circular cross-sections seem more interesting than the single points or empty set, and for the cylinder, the circles and ovals seem more interesting than the lines. We refer to these lines, single points and the empty set as **degenerate** cross-sections. We usually ignore them, concentrating instead on the interesting cross-sections – the **non-degenerate** ones.

Next comes the main example of this section – slicing a hollow 'double cone'. A double cone consists of two infinitely long cones (shaped like ice-cream cornets), joined at their pointed ends, giving the surface shown in Figure 1.4. Such cones are **right circular cones**, in the sense that the cross-sections obtained by slicing the double cone with planes at right-angles to the axis of the double cone are circles.

Double cones are often drawn with the axis vertical.

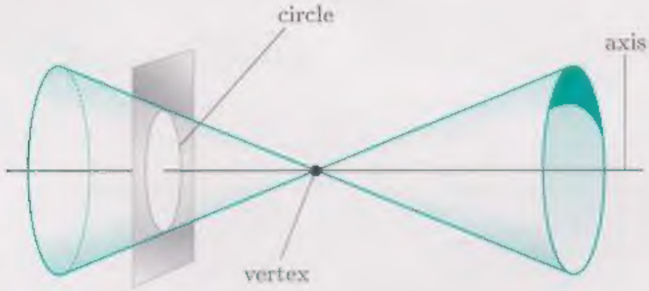


Figure 1.4 Double cone, with axis horizontal

The point at which the two cones meet is called the **vertex**, or **apex**, of the double cone.

What other cross-sections can be obtained? If you slice the double cone with planes that are not perpendicular to the axis, what curves can you obtain? In the next activity, you are asked to consider this question.

Activity 1.2 Slicing a double cone

Spend a few minutes thinking about the curves that can arise as cross-sections of a double cone. What degenerate cross-sections are there?

After trying this activity, read the solution in the text that follows.

By positioning the slicing plane in various ways, we can obtain curves of different types, as follows.

Ellipse

If we start with a vertical slicing plane, perpendicular to the axis, and tilt it slightly, then we obtain an oval curve, called an **ellipse**; see Figure 1.5. The more we tilt the plane, the more 'elongated' is the ellipse.

Your intuition may suggest that the ellipse is more pointed at one end than the other, but actually it has the same shape at both ends.

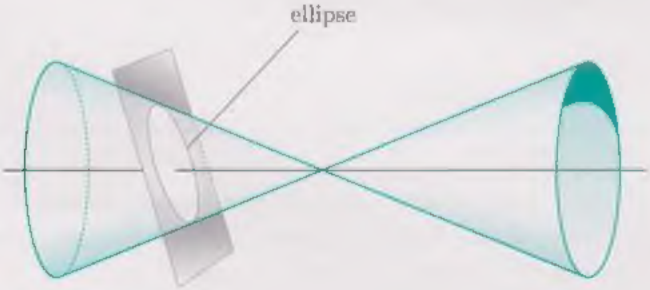


Figure 1.5 Ellipse as a cross-section

Parabola

If we continue to tilt the plane, then we reach a situation in which the slicing plane has the same slope as a line in the surface of the cone, but does not pass through the vertex; see Figure 1.6. In this case, we obtain a curve called a **parabola**, which 'extends to infinity'.

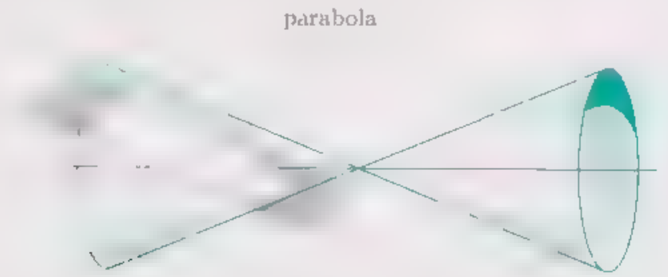


Figure 1.6 Parabola as a cross-section

Hyperbola

If we now tilt the plane further, then the slicing plane meets the double cone in a curve that is in two parts (one to the right of the vertex and one to the left of it), called a **hyperbola**; see Figure 1.7. Both parts 'extend to infinity'.

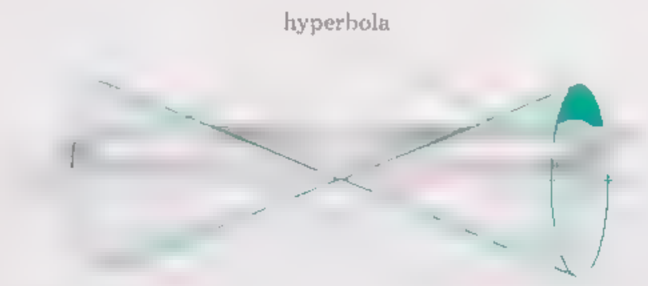


Figure 1.7 Hyperbola as a cross-section

[illegible] **conic sections** [illegible] **conics**, [illegible] types of curves. There are also some degenerate cross-sections, obtained when the slicing plane passes through the vertex of the double cone.

[illegible] only ellipses, hyperbolas and parabolas arise as non-degenerate cross-sections of the double cone.

Degenerate conics

Suppose that the slicing plane passes through the vertex of a horizontal double cone. If the plane is vertical, and perpendicular to the axis, then the cross-section is a single point (the vertex). If the slicing plane is horizontal, or nearly so, then the cross-section consists of two lines intersecting at the vertex. If the slicing plane is chosen suitably between these extremes, then the cross-section can consist of a single line through the vertex.

From now on, we shall ignore the degenerate conics and concentrate on the **non-degenerate** ones – the ellipse, parabola and hyperbola.

Visualising conics

Various methods of visualising and representing conics are explored on the Video.

Now watch Video Band A(iii), 'Visualising conics'.

1.2 Conics in science

This subsection will not be assessed.

In Section 3, some applications of conics are given.

The Ancient Greeks studied conics without knowing about their many occurrences and uses, found many centuries later. In this subsection, a few occurrences of conics as the (approximate) paths of moving objects are discussed briefly.

Motion of planets and comets

Kepler (1571–1630) was a German mathematician, physicist and astronomer.

Newton (1642–1727) was one of the greatest physicists and mathematicians of all time. His work on gravity, *Principia Mathematica*, was published in 1687.

In 1609, Johannes Kepler stated that the planets in our solar system move around the Sun in elliptical orbits. He came to this conclusion after studying the astronomical data then available, in particular those for the orbit of Mars. Some eighty years later, Isaac Newton used the new methods of 'calculus' to show that such elliptical orbits can be explained by assuming an 'inverse square law of gravitation', in which the force of attraction between the Sun and each planet is inversely proportional to the square of the distance between them. Newton claimed, moreover, that such gravitational attraction must exist between all bodies in the universe. According to Newton's theory, any object moving under the gravitational attraction of the Sun alone has an orbit which is a conic.

Halley (1656–1742) was an English mathematician and astronomer.

Further evidence for Newton's theory came when Edmond Halley correctly predicted the return in 1758 of what is now called Halley's comet, by assuming that it followed an elliptical orbit, returning close to the Sun every 76 years: see Figure 1.8.

The Sun lies at a point called the *focus* of such an elliptical orbit. See Section 3 for the definition of focus.

Figure 1.8 The orbit of Halley's comet, and that of the Earth

Motion under the influence of the Earth's gravity

Galileo (1564–1642) was an Italian astronomer and mathematician whose important work included the discovery of the use of a pendulum for measuring time and the law of acceleration for a falling object. He also developed the first astronomical telescope.

In 1638, Galileo Galilei wrote an important treatise on mechanics, called *Discourses and mathematical demonstrations concerning two new sciences*. One of the topics covered was motion under the Earth's gravity: if you throw a ball to someone else, what path does it follow?

Galileo showed that, if we ignore factors other than gravity, such as air resistance, then the path traced is part of a parabola, as shown in Figure 1.9.

Figure 1.9 Path of a ball

Motion of subatomic particles

In the early twentieth century, Ernest Rutherford showed that the path of an alpha particle moving near the nucleus of an atom is a hyperbola, as shown in Figure 1.10. Such a path occurs because the particle experiences an electrostatic repulsive force given by an inverse square law.

Rutherford (1871–1937) was a physicist from New Zealand who developed the modern concept of the atom. He won the Nobel Prize for Chemistry in 1908.

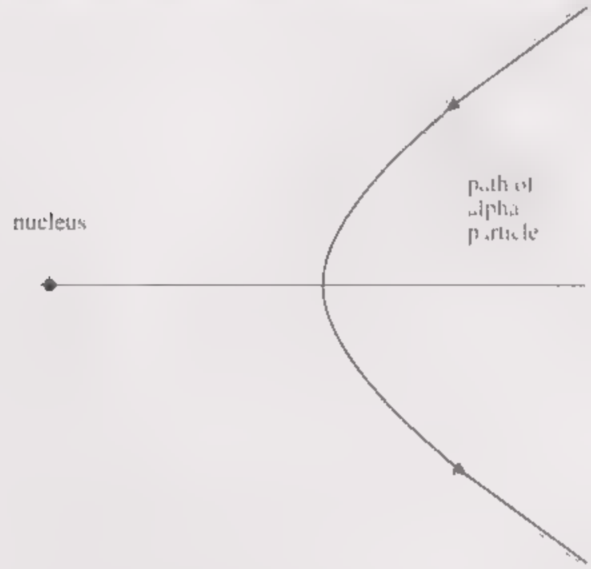


Figure 1.10 Path of an alpha particle (after Rutherford)

Summary of Section 1

This section has introduced:

- the three types of non-degenerate conic section – *ellipse*, *parabola* and *hyperbola* – as cross-sections of a double cone;
- various types of degenerate conic (pair of intersecting lines, a single line, a single point, the empty set);
- various situations in which the three types of non-degenerate conic occur.

Exercises for Section 1

There are no exercises for this section.

2 Conics in standard position

Section 1 described the types of curves, called conics, which are obtained by slicing through a double cone with a plane. In this section, algebra is used to study the three types of non-degenerate conic – the ellipse, the parabola and the hyperbola – in more detail. To do this, we need to derive equations that represent these conics. Such equations can be obtained by:

- introducing a suitable (x, y)-coordinate system in the slicing plane;
- writing down the algebraic condition which a point of the slicing plane must satisfy if it lies on the double cone.

The details of this derivation are complicated and not very illuminating! Here the equations of the three types of non-degenerate conic that result from such a derivation are simply presented and then these equations are used to obtain properties of these conics algebraically.

Fermat (1601-1665) was a French lawyer who made major contributions to mathematics, particularly number theory. Wallis (1616-1703) was an English mathematician, whose work on 'infinite arithmetic' influenced Newton in his discovery of calculus.

Amongst the first people to study conics in this way, using algebra rather than geometry, were Pierre de Fermat and John Wallis. Wallis' book *On conic sections*, published in 1655, gave a derivation of the equations of the cross-sections of a cone and also, incidentally, introduced the symbol ∞ for infinity. Fermat's book *Introduction to plane and solid loci*, published in 1679, contained many algebraic results about conics.

2.1 Ellipse

An ellipse is obtained when the slicing plane is steeper than the surface of the cone; see Figure 1.5. It can be visualised as an oval curve, and we might expect its equation to resemble that of a circle.

It turns out that the equation of an ellipse with respect to a suitable (x, y)-coordinate system in the slicing plane can be expressed in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{2.1}$$

The case $a = b$ corresponds to the circle which has centre $(0, 0)$ and radius a

$$x^2 + y^2 = a^2.$$

The case $a < b$ also gives an ellipse, but not in standard position. See Activity 2.1(b).

Here a and b are positive constants whose values depend on the slopes of the surface of the cone and of the slicing plane. With the assumption that $a \geq b$, equation (2.1) is called the equation of an **ellipse in standard position**. The constants a and b determine the shape of the ellipse.

What does equation (2.1) tell us about the shape of an ellipse? Consider the special case in which $a = 3$ and $b = 2$:

$$\frac{x^2}{9} + \frac{y^2}{4} = 1. \tag{2.2}$$

First, we check where this ellipse crosses the axes.

- The x-axis has equation $y = 0$, so points of the ellipse on the x-axis satisfy

$$\frac{x^2}{9} = 1, \quad \text{that is,} \quad x = 3 \text{ and } x = -3.$$

- The y-axis has equation $x = 0$, so points of the ellipse on the y-axis satisfy

$$\frac{y^2}{4} = 1, \quad \text{that is,} \quad y = 2 \text{ and } y = -2.$$

Thus the only points of the ellipse on the axes are $(3, 0)$, $(-3, 0)$, $(0, 2)$, and $(0, -2)$, as shown in Figure 2.1.

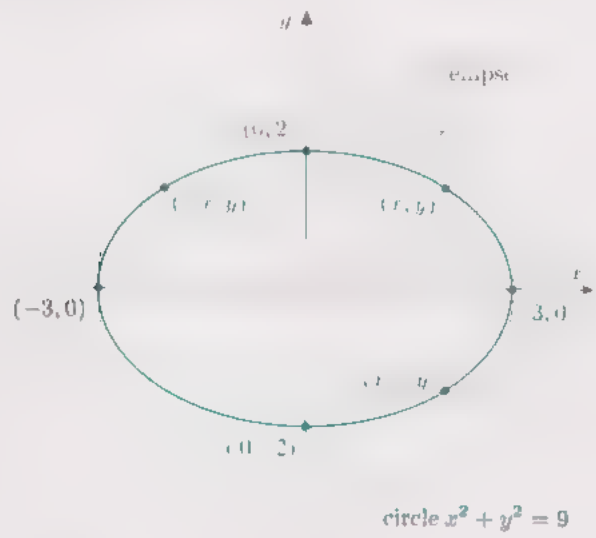


Figure 2.1 The ellipse $x^2/9 + y^2/4 = 1$

Other features of the ellipse are shown in Figure 2.1. First, we observe that the ellipse is symmetric in both axes. Expressed algebraically, this means that if the point (x, y) lies on the ellipse, then so do $(x, -y)$ and $(-x, y)$, as shown in Figure 2.1. For example, the point $(x, -y)$ lies on the ellipse because

These axes are often called the *axes of symmetry* of the ellipse.

$$\frac{x^2}{9} + \frac{(-y)^2}{4} = \frac{x^2}{9} + \frac{y^2}{4} = 1.$$

We have

$$(-y)^2 = (-y)(-y) = y^2.$$

Next, we rearrange equation (2.2) in the form

$$x^2 + \tfrac{9}{4}y^2 = 9. \qquad (2.3)$$

Equation (2.3) resembles the equation $x^2 + y^2 = 9$, which represents the circle with centre $(0, 0)$ and radius 3, also shown in Figure 2.1. A *geometric* connection between the ellipse and this circle can be obtained as follows. If (x, y) is a point on the ellipse, then, by equation (2.3),

$$x^2 + \left(\tfrac{3}{2}y\right)^2 = 9.$$

This equation tells us that the point $(x, \frac{3}{2}y)$ lies on the circle in Figure 2.1. Thus the circle can be obtained from the ellipse by

> keeping the x-coordinate of each point on the ellipse unchanged and stretching the corresponding y-coordinate by the factor $\frac{3}{2}$.

Equivalently, the ellipse can be obtained from the circle by

> keeping the x-coordinate of each point on the circle unchanged and shrinking the corresponding y-coordinate by the factor $(\frac{3}{2})^{-1} = \frac{2}{3}$.

This tells us that the ellipse is just the circle flattened by the factor $\frac{2}{3}$ in the vertical direction.

Shrinking and stretching by a given factor are both referred to as *scaling* by that factor.

Similar reasoning can be applied to equation (2.1), to give the basic properties of a general ellipse in standard position.

Basic properties of an ellipse in standard position

The ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(where } a \geq b > 0\text{)},$$

- ◇ meets the axes at the points $(a, 0)$, $(-a, 0)$, $(0, b)$ and $(0, -b)$;
- ◇ is symmetric in both the x- and y-axes;
- ◇ can be obtained from the circle $x^2 + y^2 = a^2$ by keeping the x-coordinate of each point on the circle unchanged and scaling the corresponding y-coordinate by the factor b/a.

In particular, both the right-hand end and the left-hand end of the ellipse have the same shape.

The ellipse can, alternatively, be obtained from the smaller circle

$$x^2 + y^2 = b^2$$

by keeping the y-coordinate of each point on the circle unchanged and scaling the corresponding x-coordinate by the factor a/b. Thus we can think of a non-circular ellipse either as a flattened circle or as an elongated circle.

The smaller b/a is, the flatter is the ellipse.

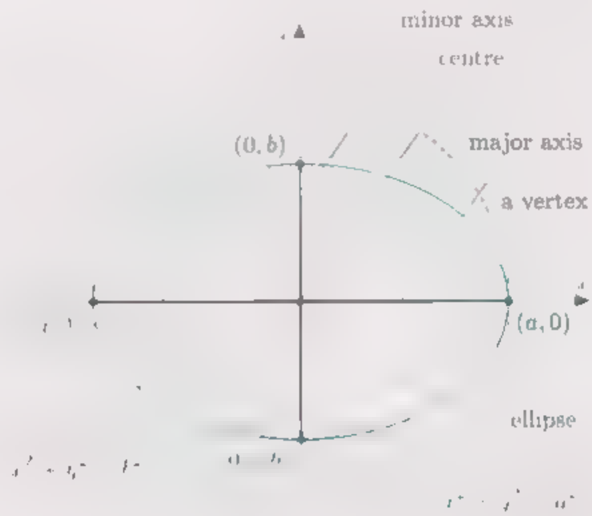


Figure 2.2 Ellipse in standard position: $x^2/a^2 + y^2/b^2 = 1$

The following notions (illustrated in Figure 2.2) are also used when describing an ellipse. The point where the axes of symmetry of an ellipse meet is called the **centre** of the ellipse; here the centre is at the origin. The points $(a, 0)$, $(-a, 0)$, $(0, b)$, $(0, -b)$ are called the **vertices** of the ellipse; the line segment from $(-a, 0)$ to $(a, 0)$ is called the **major axis** and the line segment from $(0, -b)$ to $(0, b)$ is called the **minor axis**. (The minor axis is never longer than the major axis.)

Whenever you are asked to sketch a conic, you should use the same scale on each axis (unless this is impracticable) to achieve a good representation.

The next activity asks you to sketch ellipses. A sketch of an ellipse in standard position should show the vertices accurately and exhibit symmetry in both axes. Overall, the shape should be a smooth oval.

Activity 2.1 Sketching ellipses

(a) Rearrange each of the following equations in the form of equation (2.1) and sketch the corresponding ellipses.

(i) $9x^2 + 16y^2 = 144$ (ii) $x^2 + 4y^2 - 4 = 0$

(b) Rearrange the equation $16x^2 + 9y^2 = 144$ in the form of equation (2.1), where $a < b$, and sketch the corresponding ellipse.

Solutions are given on page 49.

2.2 Hyperbola

A hyperbola is obtained when the slicing plane is less steep than the surface of the double cone: see Figure 1.7. It is in two pieces, each unbounded.

A figure is **unbounded** if there is no circle within which it lies.

It turns out that the equation of a hyperbola with respect to a suitable (x, y)-coordinate system in the slicing plane can be expressed in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{2.4}$$

Here a and b are positive numbers, whose values depend on the slopes of the surface of the cone and of the slicing plane. Equation (2.4) is called the equation of a **hyperbola in standard position**. The constants a and b determine the shape of the hyperbola.

There is no requirement here that $a \geq b$.

Consider the special case $a = 3$ and $b = 2$ in equation (2.4):

$$\frac{x^2}{9} - \frac{y^2}{4} = 1. \tag{2.5}$$

First, we check where this hyperbola crosses the axes.

- The x-axis has equation $y = 0$, so points of the hyperbola on the x-axis satisfy

  $$\frac{x^2}{9} = 1; \quad \text{that is,} \quad x = 3 \text{ and } x = -3.$$

- The y-axis has equation $x = 0$, so points of the hyperbola on the y-axis satisfy

  $$-\frac{y^2}{4} = 1; \quad \text{that is,} \quad y^2 = -4,$$

  which has no real solutions.

Thus the only points of the hyperbola on the axes are $(3, 0)$ and $(-3, 0)$, as shown in Figure 2.3.

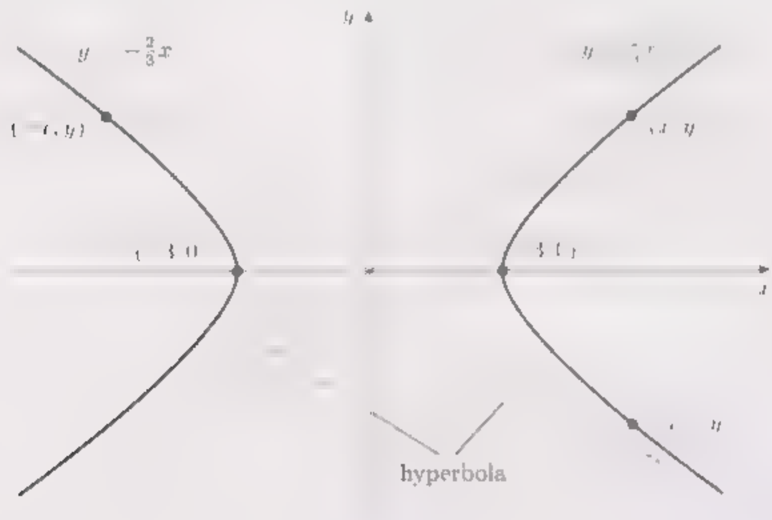


Figure 2.3 The hyperbola $x^2/9 - y^2/4 = 1$

The axes of symmetry of the hyperbola are the same as those of the ellipse.

The hyperbola, just like the ellipse, is symmetric in both the x- and y-axes. To obtain the other features shown in Figure 2.3, we first rearrange equation (2.5) in the form

$$y^2 = \tfrac{4}{9}(x^2 - 9). \tag{2.6}$$

We now solve equation (2.6) for y in terms of x, since any solutions will correspond to points on the hyperbola. For simplicity, consider first the part of the hyperbola for which $x \geq 0$.

◇ If $x > 3$, then $x^2 - 9 > 0$ and so there are two solutions

$$y = \pm\tfrac{2}{3}\sqrt{x^2 - 9}.$$

Hence there are two corresponding points on the hyperbola with coordinates $(x, \frac{2}{3}\sqrt{x^2 - 9})$ and $(x, -\frac{2}{3}\sqrt{x^2 - 9})$. So the hyperbola has two arms extending to the right symmetrically above and below the x-axis.

◇ If $x = 3$, then $x^2 - 9 = 0$ and so there is just one solution $y = 0$. This corresponds to the point $(3, 0)$, where the hyperbola cuts the x-axis.

◇ If $0 \leq x < 3$, then $x^2 - 9 < 0$ and so there are no real solutions. Hence there are no points on the hyperbola with x-coordinate satisfying $0 \leq x < 3$.

As x increases, the expression $\frac{2}{3}\sqrt{x^2 - 9}$ also increases, so the upper and lower arms of the right part of the hyperbola rise and fall, respectively. Also, $x^2 - 9 < x^2$, so

$$\tfrac{2}{3}\sqrt{x^2 - 9} < \tfrac{2}{3}\sqrt{x^2} = \tfrac{2}{3}x, \quad \text{for } x > 3.$$

Thus the upper arm of the right part lies below the line $y = \frac{2}{3}x$. It can also be shown that

$$\tfrac{2}{3}\sqrt{x^2 - 9} \text{ is close to } \tfrac{2}{3}x, \quad \text{for large values of } x.$$

For example: when $x = 99$,

$$\tfrac{2}{3}\sqrt{99^2 - 9} \simeq 65.97$$

and

$$\tfrac{2}{3}x = 66.$$

so the line $y = \frac{2}{3}x$ is a good approximation to the upper arm of the right part of the hyperbola, as shown in Figure 2.3. Similarly, the line $y = -\frac{2}{3}x$ is a good approximation to the lower arm of the right part of the hyperbola. We call such a line, which a curve approaches arbitrarily closely far from the origin, an **asymptote** of the curve.

These observations explain the shape of the right part, or **branch**, of the hyperbola in Figure 2.3. The shape of the left branch follows from the symmetry of the hyperbola in the y-axis.

Similar reasoning can be applied to equation (2.4) to give the basic properties of a general hyperbola in standard position. The form of the asymptotes can be deduced by writing equation (2.4) in the form

$$y = \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}}.$$

Basic properties of a hyperbola in standard position

The hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{(where } a, b > 0\text{)},$$

- meets the x-axis at the points $(a, 0)$ and $(-a, 0)$, and does not meet the y-axis;
- is symmetric in both the x- and y-axes;
- is in two unbounded branches, one to the right of the line $x = a$ and the other to the left of the line $x = -a$, both with asymptotes $y = \pm(b/a)x$.

The smaller b/a is, the thinner is the hyperbola.

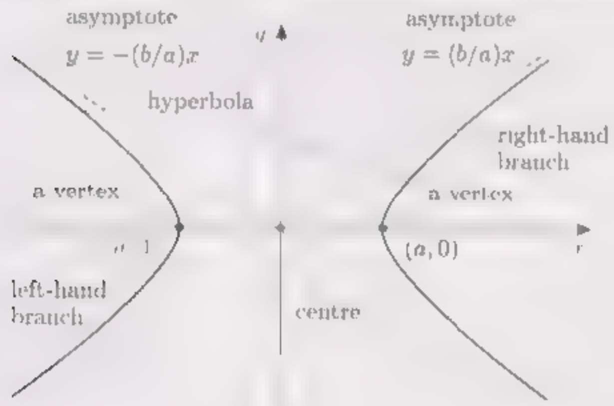


Figure 2.4 Hyperbola in standard position: $x^2/a^2 - y^2/b^2 = 1$

The points $(a, 0)$ and $(-a, 0)$ are called the **vertices** and the intersection of the axes of symmetry, which is at the origin, is called the **centre** of this hyperbola. If $b = a$, then the asymptotes are $y = \pm x$, and the curve is called a **rectangular hyperbola** because the asymptotes are perpendicular.

The next activity asks you to sketch two hyperbolas. A sketch of a hyperbola in standard position should show the vertices and asymptotes accurately, and exhibit the symmetry of the branches. The shape should be smooth, and approach the asymptotes for large values of x.

Activity 2.2 Sketching hyperbolas

Rearrange each of the following equations in the form of equation (2.4) and sketch the corresponding hyperbolas (showing the asymptotes).

(a) $9x^2 - 16y^2 = 144$ (b) $4x^2 - 3y^2 = 1$

Solutions are given on page 50.

2.3 Parabola

A parabola is obtained when the slicing plane has the same slope as the surface of the double cone; see Figure 1.6. It is in one unbounded piece.

It turns out that the equation of a parabola with respect to a suitable (x, y)-coordinate system in the slicing plane can be expressed in the form

$$y^2 = 4ax. \tag{2.7}$$

Here a is a positive number, whose value depends on the slope of the surface of the cone. You may find this equation slightly curious, since you have probably seen the U-shaped curve $y = x^2$ described as a parabola. Equation (2.7) involves y^2 and x, so the roles of the x- and y-axes are reversed. Also, there seems no good reason, at first sight, to include the factor 4 on the right. Nevertheless, it is normal to take equation (2.7) as the equation of a **parabola in standard position**. The constant a determines the shape of the parabola.

The reason for this choice of standard position will be given in Section 3.

Consider the special case $a = 1$ in equation (2.7):

$$y^2 = 4x. \tag{2.8}$$

First, the only point of this parabola on the axes is $(0, 0)$, as shown in Figure 2.5. This is because if $y = 0$, then $x = 0$, and vice versa.

Figure 2.5 The parabola $y^2 = 4x$

The parabola is symmetric in the x-axis (because $(x, -y)$ satisfies equation (2.8) whenever (x, y) does), but it is not symmetric in the y-axis.

For example, the point $(1, 2)$ lies on $y^2 = 4x$, as does $(1, -2)$, but $(-1, 2)$ does not.

To describe the other features shown in Figure 2.5, we solve equation (2.8) for y in terms of x. Any solutions will correspond to points on the parabola.

◇ If $x > 0$, then there are two solutions

$$y = \pm 2\sqrt{x}.$$

In this case, there are two corresponding points on the parabola, with coordinates $(x, 2\sqrt{x})$ and $(x, -2\sqrt{x})$, so the parabola has two arms

extending to the right symmetrically above and below the x-axis. In particular, the points $(1, 2)$ and $(1, -2)$, corresponding to $x = 1$, both lie on the parabola.

- If $x = 0$, then there is just one solution $y = 0$, corresponding to the point $(0, 0)$.
- If $x < 0$, then there are no real solutions, and hence no points on the parabola with x-coordinate satisfying $x < 0$.

As x increases, the expression $2\sqrt{x}$ also increases, so the upper and lower arms of the parabola rise and fall, respectively. However, the curve does not have asymptotes because the arms flatten out to the right. These observations explain the shape of the parabola in Figure 2.5.

Similar reasoning can be applied to equation (2.7), to give the following basic properties of a general parabola in standard position.

Basic properties of a parabola in standard position

The parabola with equation

$$y^2 = 4ax \quad (\text{where } a > 0),$$

- meets the axes at the point $(0, 0)$;
- is symmetric in the x-axis and includes the points $(a, 2a)$ and $(a, -2a)$;
- is in one unbounded part with no asymptotes.

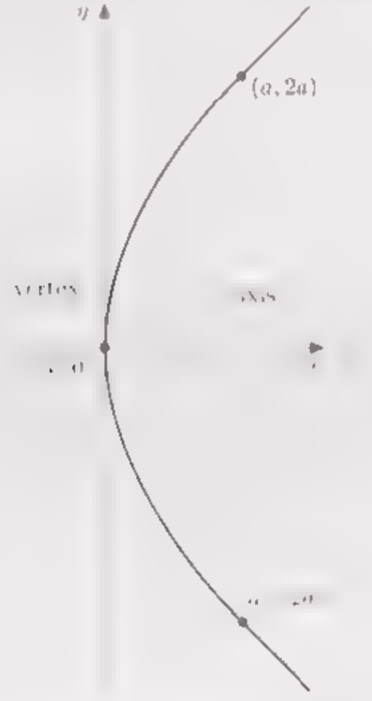


Figure 2.6 Parabola in standard position: $y^2 = 4ax$

The point where the two arms of the parabola meet, which is $(0, 0)$, is called the **vertex** of the parabola, and the x-axis is called the **axis** of the parabola. The parabola has no centre.

The next activity asks you to sketch two parabolas. A sketch of a parabola in standard position should show the vertex $(0, 0)$ and the points $(a, 2a)$ and $(a, -2a)$ accurately, and exhibit symmetry in the x-axis. Overall, the shape should be smooth.

Activity 2.3 Sketching parabolas

Rearrange each of the following equations in the form of equation (2.7) and sketch the corresponding parabolas.

(a) $y^2 = x$ (b) $4y^2 - 2x = 0$

Solutions are given on page 50.

Summary of Section 2

This section has introduced:

◇ the basic properties of the ellipse in standard position

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(where } a \geq b > 0\text{);}$$

◇ the basic properties of the hyperbola in standard position

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{(where } a, b > 0\text{);}$$

◇ the basic properties of the parabola in standard position

$$y^2 = 4ax \quad \text{(where } a > 0\text{).}$$

Exercises for Section 2

Exercise 2.1

Rearrange each of the following equations in the form of equation (2.1) and sketch the corresponding ellipses.

(a) $3x^2 + 4y^2 = 192$ (b) $x^2 + 18y^2 - 9 = 0$

Exercise 2.2

Rearrange each of the following equations in the form of equation (2.4) and sketch the corresponding hyperbola.

(a) $x^2 - 9 - 9y^2 = 0$ (b) $4y^2 - 25x^2 + 1 = 0$

Exercise 2.3

Rearrange the following equation in the form of equation (2.7) and sketch the corresponding parabola.

$$6x - 2y^2 = 0.$$

3 The focus–directrix property

In Section 2, you saw an algebraic characterisation of non-degenerate conics, namely, their equations in standard position. The circle, which is a special case of the ellipse, is defined *geometrically* as the set of points which are a fixed distance from a given point. In this section, you will see that all the non-degenerate conics have a somewhat similar geometric definition. This leads to other geometric properties of these conics, which have useful applications.

In this section, frequent use is made of the formula for the distance between one point $P_1(x_1, y_1)$ and another point $P_2(x_2, y_2)$:

The notation $P_1(x_1, y_1)$, for example, means 'the point P_1 with coordinates (x_1, y_1)'.

$$(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

from which

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Figure 3.1 illustrates the use of the notation Pd to denote the distance (that is, the *shortest* distance) from a point P to a line d, obtained by drawing a perpendicular line from the point P to the line d.

It will shortly become clear why the letter d is used for a line here.

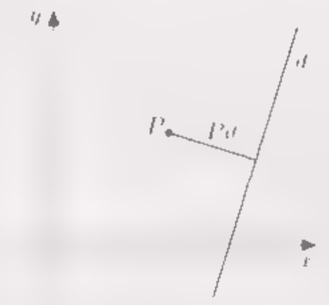


Figure 3.1 Distance from a point to a line

3.1 Focus, directrix and eccentricity

We consider each of the non-degenerate conics in turn, starting with the parabola.

Parabola

Consider the parabola $y^2 = 4x$, discussed in Section 2.

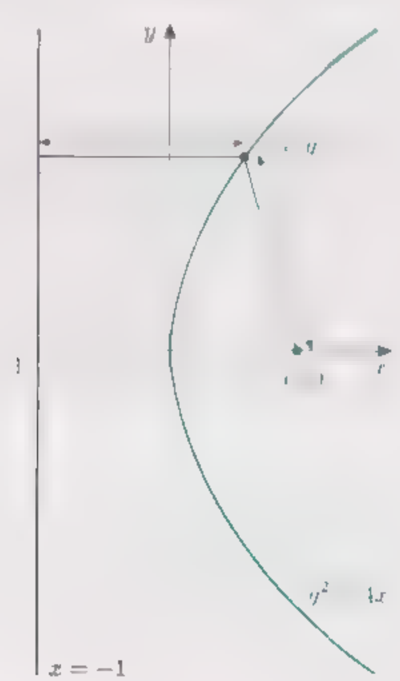


Figure 3.2 The parabola $y^2 = 4x$

For a point (x, y) lying on this parabola, we calculate the following two distances, shown in Figure 3.2.

- The distance from (x, y) to the point $(1, 0)$ is
$$\sqrt{(x-1)^2+y^2} = \sqrt{x^2-2x+1+y^2}.$$
- The distance from (x, y) to the vertical line $x = -1$ is
$$x+1.$$

The formulas for these two distances are different, but remember that (x, y) lies on the parabola, so $y^2 = 4x$. Using this equation in the first formula, the distance from (x, y) to $(1, 0)$ is equal to:

$$\begin{aligned}\sqrt{x^2-2x+1+y^2} &= \sqrt{x^2-2x+1+4x} \\ &= \sqrt{x^2+2x+1} \\ &= \sqrt{(x+1)^2} = x+1,\end{aligned}$$

The equal distances property is straightforward to check at the points $(0, 0)$, $(1, 2)$ and $(1, -2)$ on the parabola.

since $x + 1 \geq 0$ when (x, y) lies on the parabola. Thus the two distances calculated above are *always* equal when (x, y) lies on the parabola. The fact that each point of the parabola is equidistant from the point $(1, 0)$ and from the line $x = -1$ is illustrated in Figure 3.3.

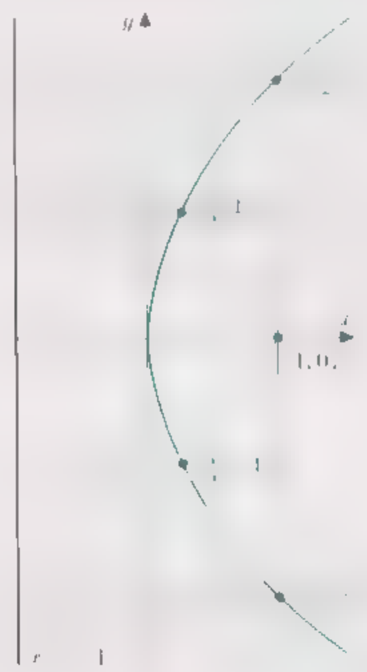


Figure 3.3 Equal distances

This is a nice property of the parabola $y^2 = 4x$, but could it be true only in this special case? In the next activity, you are asked to check that, with appropriate choices of the point and the line, *all* parabolas in standard position have this property.

Activity 3.1 Generalising the equal distances property

Let (x, y) be a point on the parabola $y^2 = 4ax$ (where $a > 0$).

(a) Write down the distance from (x, y) to the point $(a, 0)$.

(b) Write down the distance from (x, y) to the line $x = -a$.

(c) Show that the distances in parts (a) and (b) are the same.

Solutions are given on page 51.

Comment

A slight modification of the solution shows that if (x, y) is a point for which the distances in parts (a) and (b) are the same, then $y^2 = 4ax$. Therefore the parabola consists of *precisely* those points whose distance to $(a, 0)$ equals their distance to the line $x = -a$.

So every parabola in standard position has the equal distances property. The point $(a, 0)$ is called the *focus* of the parabola and the line $x = -a$ is called its *directrix*. These are shown in Figure 3.4.

The name focus (Latin for 'hearthside') derives from an optical property of the parabola, discussed later; it was introduced by Kepler.

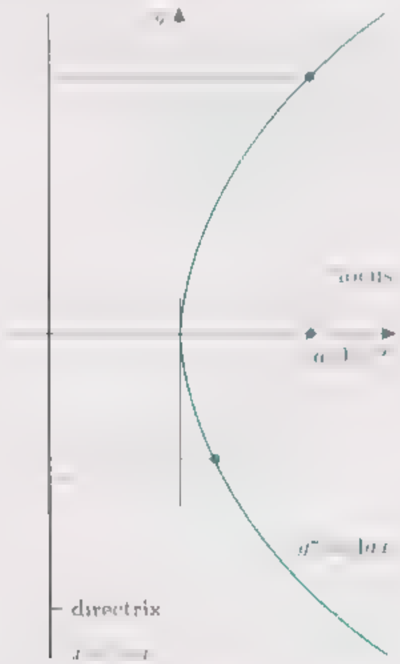


Figure 3.4 Focus–directrix property of the parabola in standard position

The number 4 in the equation $y^2 = 4ax$ is included so that the focus and directrix of this parabola have simple forms.

Activity 3.2 Finding the focus and directrix

For each of the following parabolas, find the focus and the directrix, and indicate these on a sketch of the parabola.

(a) $y^2 = x$ (b) $4y^2 - 2x = 0$

Solutions are given on page 51.

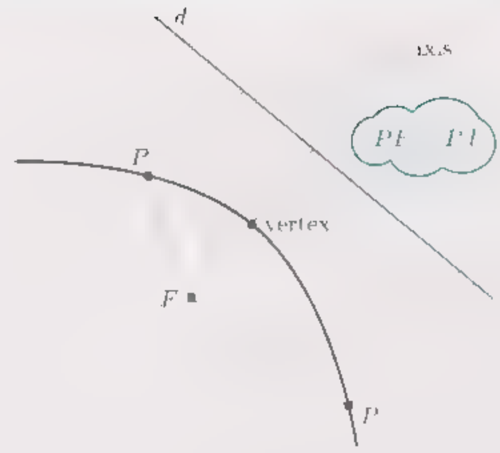
You sketched these parabolas in Activity 2.3.

This **focus–directrix property** gives a geometric definition of a **parabola in general position**, illustrated in Figure 3.5, overleaf.

Focus–directrix definition of the parabola
Let F be a point and d a line not passing through F. Then the set of points P satisfying

$$PF = Pd$$

is a parabola with **focus** F and **directrix** d.



Note that the parabola and its focus are both on the same side of the directrix.

Figure 3.5 Focus–directrix definition of the parabola

In this definition, the point F and the line d can be anywhere in the plane, as long as F does not lie on d. The resulting parabola is congruent to some parabola in standard position. The axis and vertex of such a parabola are defined in a similar way to the parabola in standard position. The line through F perpendicular to d is the **axis** and the point of the parabola on this axis is the **vertex**.

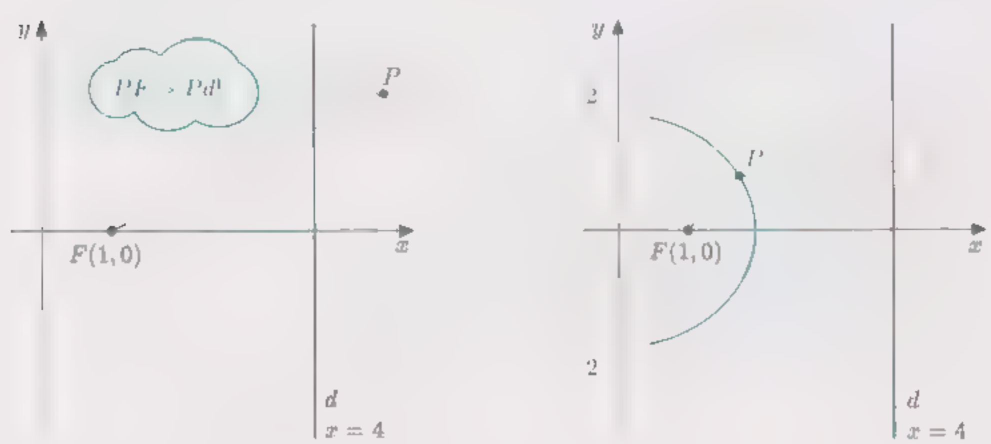
Two figures are congruent if they have the same shape and size. Some examples of congruent curves are given in Section 4.

Next, we explore what happens if the 'equal distances' property $PF = Pd$ is replaced by a 'proportional distances' property. It turns out that the ellipse and hyperbola arise in this way.

Ellipse

We begin by considering a special case. Let F be the point with coordinates $(1, 0)$ and let d be the line $x = 4$; see Figure 3.6. What is the equation of the curve on which a general point $P(x, y)$ satisfies

$$PF = \tfrac{1}{2}Pd? \tag{3.1}$$



(a) P to right of d?

(b) P lies to left of d

Figure 3.6 $PF = \frac{1}{2}Pd$

First notice that if P lies on or to the right of the line d, then PF is greater than Pd, so equation (3.1) cannot hold. Therefore we may assume that P lies to the left of the line d. With this assumption, we have $Pd = 4 - x$ and

$$PF = \sqrt{(x-1)^2 + y^2}$$

Thus equation (3.1) can be written in the form

$$[illegible]$$

On squaring both sides of this equation, we obtain

$$(x-1)^2 + y^2 = \tfrac{1}{4}(4-x)^2,$$

and further rearrangements lead to

$$4(x^2 - 2x + 1 + y^2) = 16 - 8x + x^2,$$

$$3x^2 + 4y^2 = 12,$$

and then

$$\frac{x^2}{4} + \frac{y^2}{3} = 1.$$

This is the equation of an ellipse in standard position, with $a = 2$ and $b = \sqrt{3}$. So this ellipse has a focus-directrix property in which 'equal distances' are replaced by 'proportional distances'. This property of the ellipse is illustrated in Figure 3.7.

A check on the working is that, for $P = (2,0)$,

$$PF = 1 = \tfrac{1}{2}Pd$$

and, for $P = (-2,0)$,

$$PF = 3 = \tfrac{1}{2}Pd$$

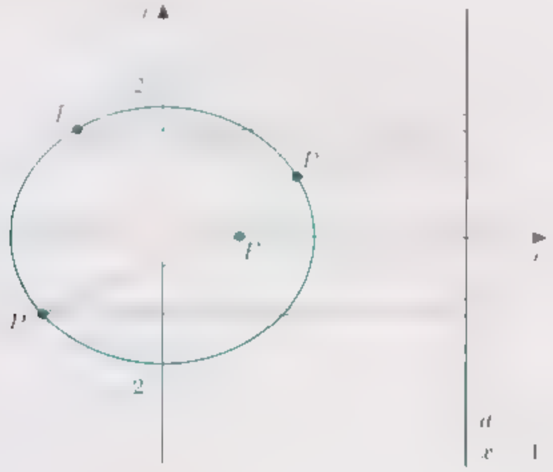


Figure 3.7 Proportional distances: $PF = \frac{1}{2}Pd$

You are now asked to verify that all ellipses in standard position have a similar focus-directrix property with proportional distances.

Activity 3.3 Generalising the proportional distances property

---

[illegible] Then let F be the point $(ae, 0)$, let d be the line $x = a/e$ and let $P(x, y)$ satisfy

$$PF = ePd.$$

Since $PF < Pd$, any such point P is closer to F than it is to the line d, and so must lie to the left of d.

Do not spend too long on this activity.

The expression ePd means $e \times Pd$.

(a) Write down expressions for PF and Pd.

(b) Use the equation $PF = ePd$ to find an equation satisfied by $P(x, y)$.

(c) Show that the equation in part (b) is that of an ellipse in standard position, with constants a and b, where $b = a\sqrt{1 - e^2}$.

(d) Express e in terms of a and b.

Solutions are given on page 51.

It follows from the solution to Activity 3.3 that if

F is the point $(ae, 0)$, d is the line $x = a/e$, and $e = \sqrt{1 - b^2/a^2}$,

then the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has the property that $PF = ePd$ when P lies on the ellipse. Once again, we call F the *focus* of the ellipse and d the *directrix*. The quantity e is called the *eccentricity* of the ellipse. This property makes sense only if $e > 0$, so we need $b < a$.

If $b = a$, then the ellipse is a circle, which does not have such a focus–directrix property.

Given the symmetry of the ellipse, you might suspect that there is another focus F' and directrix d'. This is indeed the case. If F' is the point $(-ae, 0)$ and d' is the line $x = -a/e$, then any point P on the ellipse satisfies both $PF = ePd$ and $PF' = ePd'$. All these features are shown in Figure 3.8.

Figure 3.8 Focus–directrix properties of the ellipse in standard position

Notice that the eccentricity $e = \sqrt{1 - b^2/a^2}$ also tells you how squashed (or elongated) the ellipse is. For a nearly circular ellipse, the quantity b/a is close to 1, so the eccentricity is close to 0. On the other hand, for a very squashed ellipse, the quantity b/a is small, so the eccentricity is close to 1.

For example, the Earth's orbit round the Sun is nearly circular, with

$e \simeq 0.0167$,

whereas that of Halley's comet is elongated, with

$e \simeq 0.967$.

Activity 3.4 Finding the foci, directrices and eccentricity

For each of the following ellipses, find the foci, directrices and eccentricity, and mark the foci and directrices on a sketch of the ellipse.

You sketched these ellipses in Activity 2.1.

(a) $9x^2 + 16y^2 = 144$

(b) $x^2 + 4y^2 - 4 = 0$

Solutions are given on page 52.

The **focus–directrix property** gives a geometric definition of the **ellipse in general position**, illustrated in Figure 3.9.

Focus–directrix definition of the ellipse

Let F be a point, d a line not passing through F and e a number such that $0 < e < 1$. Then the set of points P satisfying

$$PF = ePd$$

is an ellipse with **focus** F, **directrix** d and **eccentricity** e.

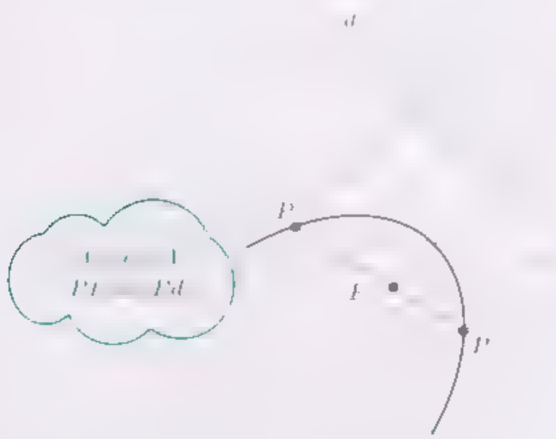


Figure 3.9 Focus–directrix definition of the ellipse

The point F and the line d can be anywhere in the plane, as long as F does not lie on d. The resulting ellipse is congruent to some ellipse in standard position. The **centre**, **vertices** and **major** and **minor axes** of such an ellipse are defined in a similar way to the ellipse in standard position.

Hyperbola

The following activity asks you to find the equation of another curve for which the 'equal distances' in the focus–directrix definition of the parabola are replaced by 'proportional distances'.

Activity 3.5 Proportional distances

Let the point F have coordinates $(2, 0)$ and let d be the line $x = 1$. Show that the equation of the curve on which a general point $P(x, y)$ satisfies

$$PF = \sqrt{2}Pd$$

is

$$\frac{x^2}{2} - \frac{y^2}{2} = 1.$$

Solutions are given on page 52.

The solution to Activity 3.5 is the equation of a hyperbola in standard position, with $a = \sqrt{2}$ and $b = \sqrt{2}$. Therefore, this hyperbola has a focus–directrix property in which 'equal distances' are replaced by 'proportional distances'. This property of the hyperbola is illustrated in Figure 3.10.

A check on the working is that, for $P(\sqrt{2}, 0)$,

$$\begin{aligned} PF &= 2 - \sqrt{2} \\ &= \sqrt{2}(\sqrt{2} - 1) \\ &= \sqrt{2}Pd \end{aligned}$$

and, for $P(-\sqrt{2}, 0)$,

$$\begin{aligned} PF &= 2 + \sqrt{2} \\ &= \sqrt{2}(\sqrt{2} + 1) \\ &= \sqrt{2}Pd. \end{aligned}$$

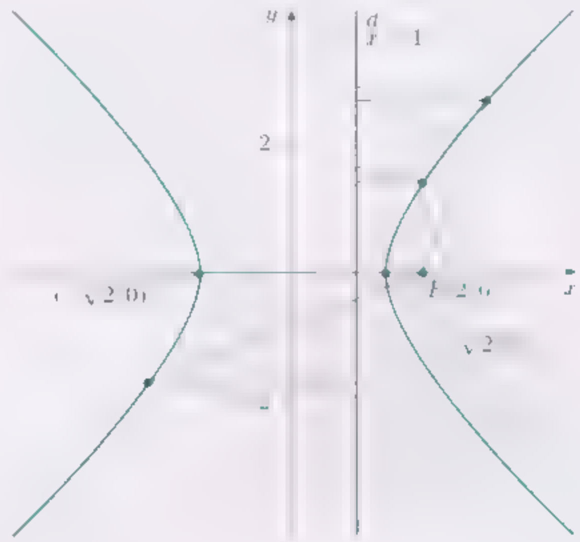


Figure 3.10 Proportional distances

In fact, all hyperbolas have a similar focus–directrix property with proportional distances. This fact can be established by using an argument similar to that in Activity 3.3, but it is omitted here.

The result is as follows. Suppose that $a > 0$ and $e > 1$. If

F is the point $(ae, 0)$, d is the line $x = a/e$, and $e = \sqrt{1 + b^2/a^2}$,

then the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has the property that $PF = ePd$, when P lies on the hyperbola. Once again, we call F the *focus* of the hyperbola and d the *directrix*. The quantity e is called the *eccentricity* of the hyperbola.

The hyperbola also has another focus F' and directrix d'. If F' is the point $(-ae, 0)$ and d' is the line $x = -a/e$, then any point P on the hyperbola satisfies both $PF = ePd$ and $PF' = ePd'$. All these features are shown in Figure 3.11.

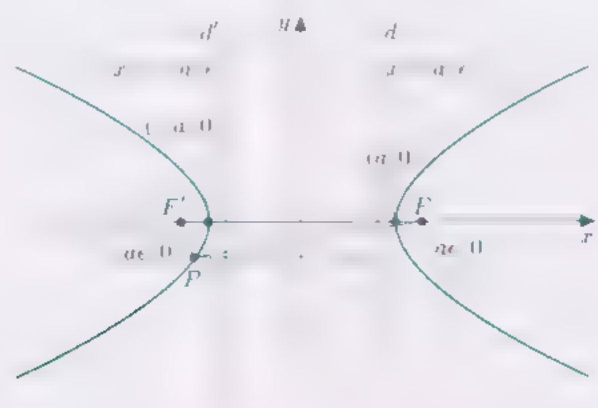


Figure 3.11 Focus–directrix properties of the hyperbola in standard position

Notice that the eccentricity also tells you how thin the hyperbola is; that is, how small is the angle between the asymptotes $y = \pm(b/a)x$. For a very thin hyperbola, the quantity b/a is very small, so the eccentricity $e = \sqrt{1 + b^2/a^2}$ is very close to 1. On the other hand, for a very fat hyperbola, the quantity b/a is very large, so the eccentricity is also very large.

Activity 3.6 Finding the foci, directrices and eccentricity

For each of the following hyperbolas, find the foci, directrices and eccentricity and mark the foci and directrices on a sketch of the hyperbola.

(a) $9x^2 - 16y^2 = 144$ (b) $4x^2 - 3y^2 = 1$

You sketched these hyperbolas in Activity 2.2.

Solutions are given on page 53.

Once again, the **focus–directrix property** gives a geometric definition of a **hyperbola in general position**, illustrated in Figure 3.12.

Focus–directrix definition of the hyperbola

Let F be a point, d a line not passing through F and e a number such that $e > 1$. Then the set of points P satisfying

$$PF = ePd$$

is a hyperbola with **focus** F, **directrix** d and **eccentricity** e.

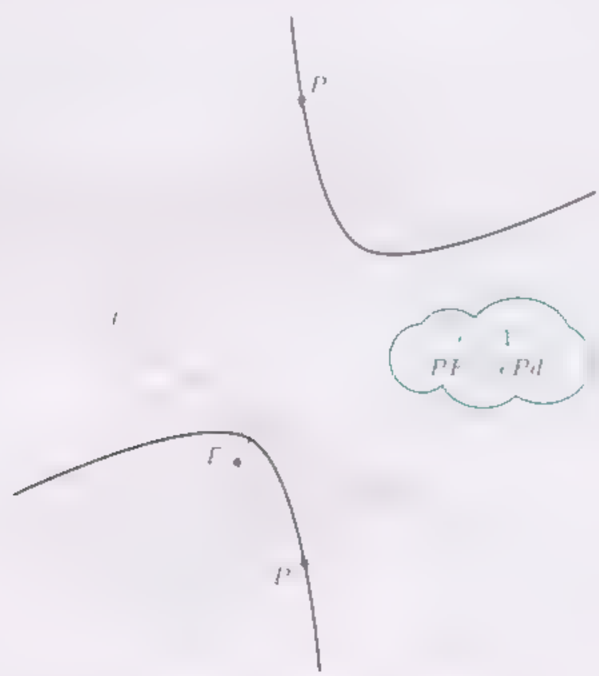


Figure 3.12 Focus–directrix definition of the hyperbola

The point F and the line d can be anywhere in the plane, as long as F does not lie on d. The resulting hyperbola is congruent to some hyperbola in standard position. The **centre** and **vertices** of such a hyperbola are defined in a similar way to the hyperbola in standard position.

The equation involved in all these focus–directrix definitions is of the form

$$PF = ePd.$$

This equation gives a parabola if $e = 1$, an ellipse if $0 < e < 1$ and a hyperbola if $e > 1$. By analogy with the ellipse and the hyperbola, it is natural to say that the parabola has **eccentricity** $e = 1$.

Some texts say that a circle has eccentricity $e = 0$, even though it does not have a focus–directrix property.

Notice that, as the eccentricity e increases from near 0 to 1 and then beyond, the corresponding conic changes from a near-circular ellipse to an elongated ellipse, then to a parabola, and finally a hyperbola, first thin and then fat. These changes are similar to the changes seen in the solution to Activity 1.2, as the slicing plane is tilted further and further.

This subsection concludes by pointing out the origin of the names of the three non-degenerate conics.

The names of the conic sections

Ellipse, *parabola* and *hyperbola* are the Greek names given to the non-degenerate conics by Apollonius in the third century BC. They corresponded to a certain area associated with these curves being *less than*, *equal to*, or *greater than* another area.

We still use related words in English: a remark is *elliptical* if it is less than clear, a *parable* is a story pointing out some similarity, and *hyperbole* is exaggeration.

3.2 Applications of conics

This section will not be assessed.

Brief indications are given here of several applications of conics. These are related to two further properties of conics, which are now described.

Two further properties

The first property concerns the sum of the distances from a point on an ellipse to the foci of the ellipse.

If P lies on the ellipse $x^2/a^2 + y^2/b^2 = 1$, with foci F and F', and directrices d and d', then

$$PF + PF' = 2a. \tag{3.2}$$

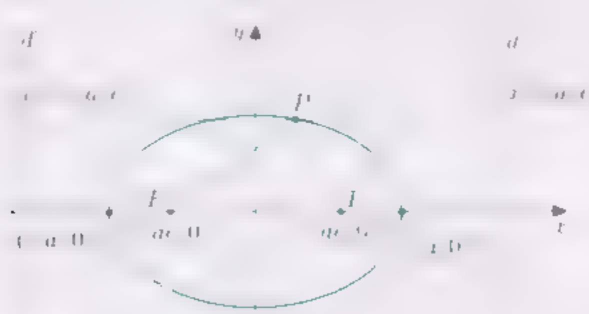


Figure 3.13 $PF + PF' = 2a$

To demonstrate this property, we use the facts that

$$PF = ePd \quad \text{and} \quad PF' = ePd',$$

where e is the eccentricity of the ellipse.

These give

$$\begin{aligned} PF + PF' &= e(Pd + Pd') \\ &= e(dd'), \end{aligned}$$

where dd' denotes the perpendicular distance between the two directrices. Since d is the line $x = a/e$ and d' is the line $x = -a/e$, this distance is $2a/e$, so

$$\begin{aligned} PF + PF' &= e(2a/e) \\ &= 2a, \end{aligned}$$

for each point P on the ellipse.

This property gives a simple method of drawing an ellipse. Pins are placed at the foci F and F', and a loop of string is placed around the pins, held taut by a pencil at P. The path of the pencil as it moves is an ellipse. This is called the *gardener's method*, since it is a convenient way to construct elliptical flower beds.

The gardener's method has been known since at least the sixth century AD, when it was described by Anthemius, the architect of the church of Hagia Sophia in Constantinople (now Istanbul).

Similar reasoning leads to a property for the hyperbola concerning the *difference* between the distances from a point on the hyperbola to the foci.

If P lies on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ with foci F and F', then

the magnitude of $(PF' - PF) = 2a$.

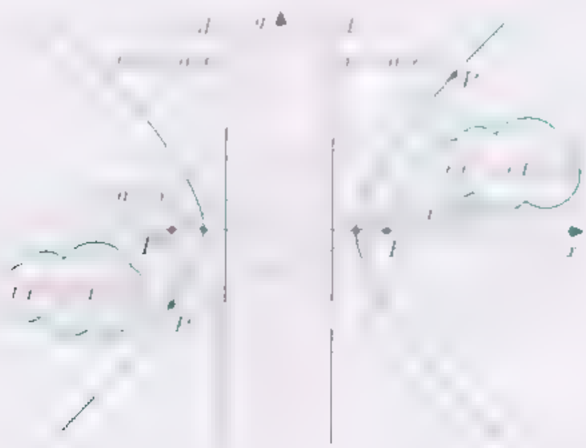


Figure 3.14 Two positions for P

This property of the hyperbola has played an important role in certain navigational systems. These systems use linked radio transmitters, which transmit signals simultaneously. The user's receiver measures the small *differences* in the arrival times of these signals and then calculates the user's position by relating these time differences to families of hyperbolas that have their foci at the transmitters.

Reflection properties of the conics

When a ray of light is reflected by a mirror, the angle between the incident ray and the mirror is equal to the angle between the reflected ray and the mirror. Using this fact, it can be shown that each of the three non-degenerate conics has a characteristic reflection property.

These properties can be proved using the focus–directrix definitions of the conics, but the details are omitted.

Reflection properties of conics

A mirror is shaped to form a conic and a ray of light, originating from a focus F of the conic, is reflected by the mirror.

- ◇ In the case of an ellipse, the ray is reflected through the other focus F'.
- ◇ In the case of a parabola, the ray is reflected parallel to the axis of the parabola.
- ◇ In the case of a hyperbola, the ray is reflected as if it came from the other focus F'.

These three reflection properties are illustrated in Figure 3.15.

Figure 3.15 Reflection properties of conics

The reflection properties of conics have several important applications. Since light rays originating from the focus of a parabolic mirror are reflected by the mirror as a beam of parallel rays, parabolic reflectors are used in searchlights. For the same reason, rays which approach a parabolic mirror parallel to its axis are reflected to pass through the focus. This reflection property is exploited in telescopes and may be combined with the reflection property of the hyperbola in the arrangement shown in Figure 3.16. Using two mirrors in this way means that the reflected light travels further before being focused, thus increasing the resolution of the final image. Also this image is positioned in a convenient location for viewing.

Diocles (second century BC) included this fact in his book *On burning mirrors.*

Most large astronomical telescopes use this arrangement, which was invented by the French scientist Cassegrain in 1672.

Figure 3.16 Parabolic and hyperbolic mirrors combined

Finally, the reflection property of the ellipse has been exploited in the construction of 'whispering galleries'. If the shape of the ceiling is obtained by rotating the upper half of an ellipse about the ellipse's major axis, then sound waves emerging from one focus are reflected by the ceiling through the other focus — moreover, all the reflected sound waves arrive simultaneously at the other focus since, by equation (3.2), they all travel the same distance!

The surface obtained by such a rotation is called an *ellipsoid*.

Summary of Section 3

This section has introduced:

◇ the focus–directrix properties of non-degenerate conics in standard position, summarised in the following table:

curve	equation	focus	directrix	eccentricity
ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (where $a>b>0$)	$(\pm ae, 0)$	$x=\pm\frac{a}{e}$	$0<e<1$, $e=\sqrt{1-b^2/a^2}$
parabola	$y^2=4ax$ (where $a>0$)	$(a,0)$	$x=-a$	$e=1$
hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (where $a,b>0$)	$(\pm ae, 0)$	$x=\pm\frac{a}{e}$	$e>1$, $e=\sqrt{1+b^2/a^2}$

The asymptotes of the hyperbola are $y=\pm(b/a)x$.

◇ the focus–directrix definition of a non-degenerate conic in general position, as the sets of points P satisfying

$$PF = ePd,$$

where F is the focus, d is the directrix and e is the eccentricity:

$$\begin{cases} 0<e<1 & \text{gives an ellipse,} \\ e=1 & \text{gives a parabola,} \\ e>1 & \text{gives a hyperbola;} \end{cases}$$

◇ some further properties of conics, and their applications.

Exercises for Section 3

Exercise 3.1

Find the focus and the directrix of the following parabola and indicate them on a sketch of the parabola. (You sketched this parabola in Exercise 2.3.)

$$6x-2y^2=0$$

Exercise 3.2

For each of the following ellipses, find the foci, directrices and eccentricity, and mark the foci and directrices on a sketch of the ellipse. (You sketched these ellipses in Exercise 2.1.)

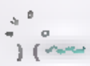
(a) $3x^2+4y^2=192$ (b) $x^2+18y^2-9=0$

Exercise 3.3

For each of the following hyperbolas, find the foci, directrices and eccentricity. (You sketched these hyperbolas in Exercise 2.2.)

(a) $x^2-9-9y^2=0$ (b) $4y^2-25x^2+1=0$

4 Quadratic curves



To study this section, you will need an audio cassette player and Audio Tape 1.

Section 1 introduced the family of *conics*, which occur as cross-sections of a double cone, and Section 2 introduced the equations of conics in standard position:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad y^2 = 4ax.$$

Another property that these conics have in common is that they are all **quadratic curves**, that is, they are represented by quadratic equations of the form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{4.1}$$

where A, B, C, D, E and F are real numbers, with A, B, C not all zero. For example, the equation of the parabola in standard position can be rearranged as $y^2 - 4ax = 0$, so it is of this form with

$$A = 0, \; B = 0, \; C = 1, \; D = -4a, \; E = 0 \text{ and } F = 0.$$

This is a significant similarity, and it suggests a question to which the answer is by no means obvious.

What types of curves are represented by equation (4.1)?

To see why the answer to this question is not obvious, consider the quadratic equations

$$x^2 + y^2 + 1 = 0 \quad \text{and} \quad xy = 0.$$

The equation $x^2 + y^2 + 1 = 0$ has no solutions because $x^2 + y^2 \geq 0$, for all x and y. So in this case the curve is the empty set.

The equation $xy = 0$ is satisfied when $x = 0$ and when $y = 0$. So in this case the curve consists of the two axes.

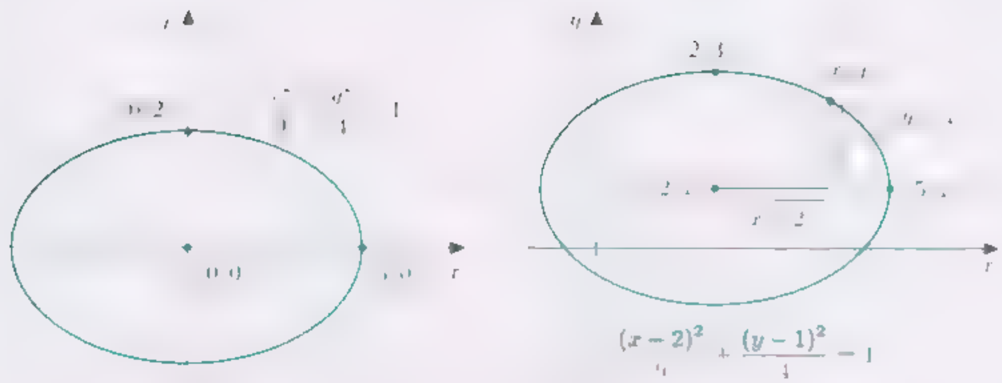
The case when $B \neq 0$ is taken up in Chapter A3.

In this section, we explore the above question mainly in the special case when $B = 0$, so equation (4.1) has no term in xy. We find that such quadratic equations often represent non-degenerate conics, though not necessarily in standard position. In particular, we meet examples of conics obtained by shifting, or *translating*, a conic in standard position; that is, moving the conic to a new position in the plane without rotating it. For example, if the ellipse $x^2/9 + y^2/4 = 1$ is translated so that its centre is at $(2, 1)$, then the horizontal and vertical displacements of a general point (x, y) on the translated ellipse from its centre are $x - 2$ and $y - 1$, respectively: see Figure 4.1. Therefore this general point (x, y) satisfies

This translated ellipse is considered again in Frame 1.

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1$$

and this is the equation of the translated ellipse.



(a) Original ellipse (b) Translated ellipse

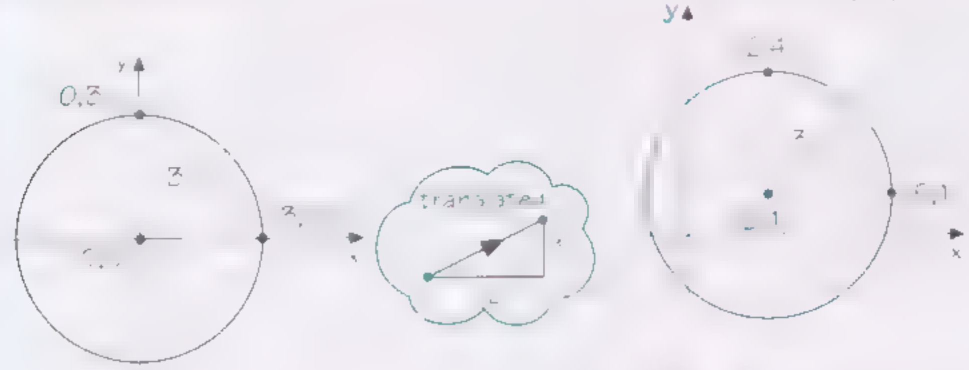
Figure 4.1 Translating an ellipse

Now listen to Audio Tape 1, Band 2, 'Quadratic curves'.

Examples of quadratic curves

Circle, centre at (0, 0)

Circle, centre at (2,1)



$$x^2 + y^2 - 4x - 2y - 4 = 0 \qquad (1)$$

Ellipse in standard position

Ellipse, centre at (2,1)

$$4x^2 + 9y^2 - 16x - 18y - 11 = 0 \qquad (2)$$

Equation of general quadratic curve:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Circle, equation (1): $A = 1, B = 0, C = 1, D = -4, E = -2, F = -4$.

Ellipse, equation (2): $A = 4, B = 0, C = 9, D = -16, E = -18, F = -11$.

Activity 4.1 Identifying coefficients

Write down the values of A, B, C, D, E and F for each of the following equations.

(a) $4x^2 - 9y^2 = 36$ (b) $y^2 = 4x$ (c) $xy = 1$

Solutions are given on page 53.

Sketching quadratic curves

Sketch the curve with equation

ellipse? parabola? hyperbola?

$$x^2 - 4y^2 - 6x - 40y - 95 = 0.$$

Collect x and y terms:

$$(x^2 - 6x) - 4(y^2 + 10y) - 95 = 0.$$

Complete the squares:

$$x^2 - 6x = (x^2 - 6x + 9) - 9$$
$$y^2 + 10y = (y^2 + 10y + 25) - 25$$

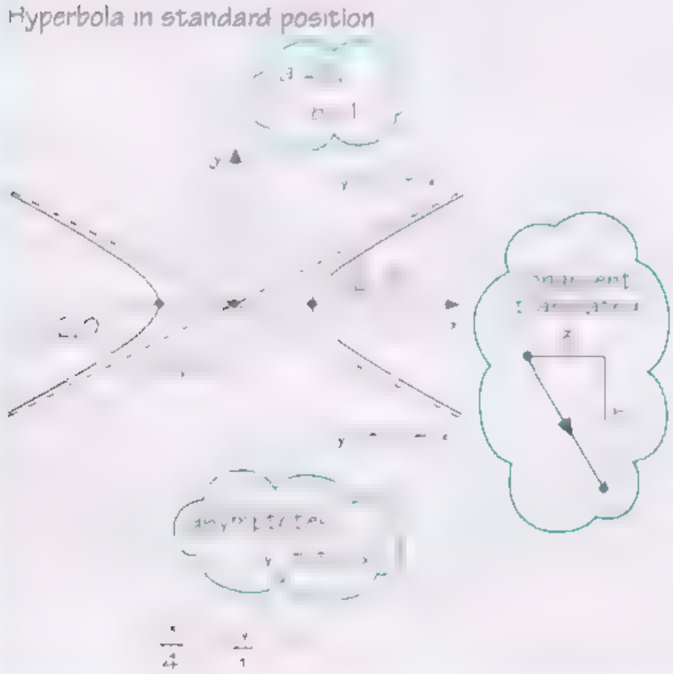
$$\left((x-3)^2 - 9\right) - 4\left((y+5)^2 - 25\right) - 95 = 0.$$

Simplify:

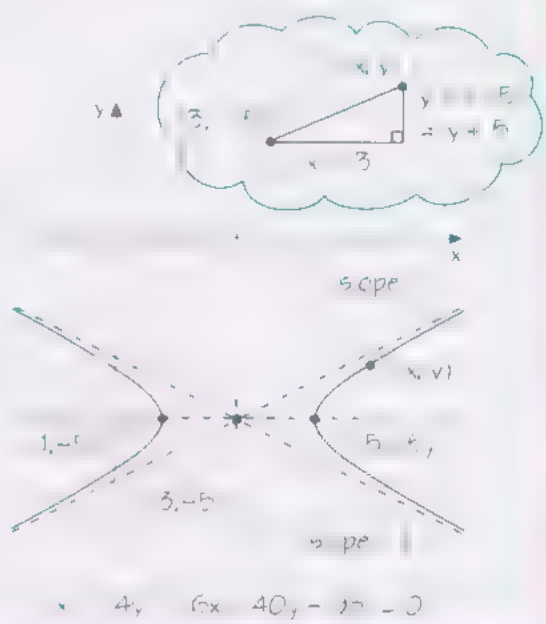
$$(x-3)^2 - 4(y+5)^2 = 4; \text{ that is,}$$

$$\boxed{\frac{(x-3)^2}{4} - \frac{(y+5)^2}{1} = 1}$$

Hyperbola in standard position



Hyperbola, centre at (3,−5)



Activity 4.2 Sketching quadratic curves

Sketch each of the following quadratic curves.

(a) $3x^2 + 4y^2 - 48x - 48y + 324 = 0$ (b) $y^2 - 16x - 8y = 0$

Solutions are given on page 53.

Summary of Section 4

This section has introduced

- quadratic curves, with equations of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

  where A, B, C are not all zero;
- a technique for showing that quadratic curves for which $B = 0$ often represent conics obtained by translating conics in standard position.

Exercises for Section 4

Exercise 4.1

Classify each of the following curves as an ellipse, parabola or hyperbola, and then sketch each curve. (Your sketch should show the vertices, axes of symmetry, and the slopes of any asymptotes.)

(a) $3x^2 - 2y^2 + 6x + 12y - 33 = 0$

(b) $x^2 + 16y^2 - 10x + 64y + 73 = 0$

5 Parametrising conics

The equations you have seen for parabolas, ellipses and hyperbolas are all quadratic equations involving the variables x and y. These equations specify the relationship between the x-coordinate and y-coordinate of any point on the conic. An alternative way to represent such curves is to give equations for x and y in terms of another variable, usually called t. These equations are chosen so that each value of t gives rise to a point (x, y) on the curve. The new variable t is called a **parameter**, sometimes thought of as 'time', and the equations for x and y in terms of t are called **parametric equations** or a **parametrisation** of the curve.

For example, the circle $x^2 + y^2 = r^2$, which has centre $(0, 0)$ and radius r, has parametric equations

$$x = r\cos t, \quad y = r\sin t \quad (0 \le t \le 2\pi).$$

See MST121, Chapter A2.

Figure 5.1 shows several points on the unit circle (for which $r = 1$), each corresponding to a value of t. The parameter t represents the angle (in radians), measured anticlockwise from the positive axis.

The fact that points (x, y) satisfy $x^2 + y^2 = 1$ on the unit circle corresponds to the equation

$$\cos^2 t + \sin^2 t = 1.$$

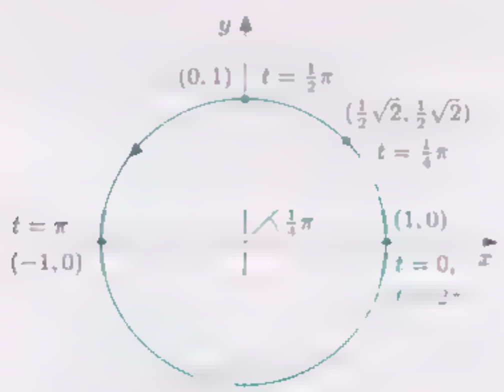


Figure 5.1 Parametrisation of the unit circle

The arrow on the circle shows the direction of motion of the point (x, y) around the circle as t increases from 0 to 2π.

Parametric equations are useful for plotting curves, particularly when using a computer. If we let the parameter t run through a sequence of values, then the parametric equations give a sequence of corresponding points (x, y) lying on the curve. In this section, parametric equations are given for the ellipse, parabola and hyperbola.

5.1 Standard parametrisations

Ellipse

The ellipse in standard position,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (\text{where } a \geq b > 0),$$

is obtained by squashing the circle $x^2+y^2=a^2$ by the factor b/a in the vertical direction. So it is natural to seek a parametrisation similar to that of the circle.

Since any point Q on the circle $x^2+y^2=a^2$ is of the form $(a\cos t, a\sin t)$, the point P vertically below Q on the ellipse must have the same x-coordinate, namely $a\cos t$, and y-coordinate $(b/a)a\sin t = b\sin t$; see Figure 5.2. This gives the **standard parametrisation of the ellipse**

$$x = a\cos t, \quad y = b\sin t \quad (0 \leq t \leq 2\pi). \tag{5.1}$$

As t increases from 0 to 2π, the point $(a\cos t, b\sin t)$ travels once round the ellipse in the direction of the arrow, starting and finishing at $(a, 0)$.

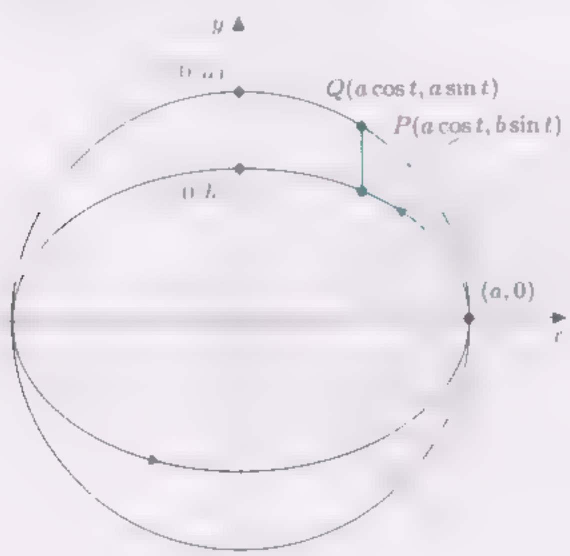


Figure 5.2 Standard parametrisation of the ellipse

As a check, note that equations (5.1) do satisfy the equation of the ellipse in standard position:

$$\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2} &= \frac{(a\cos t)^2}{a^2}+\frac{(b\sin t)^2}{b^2}\\ &= \cos^2 t+\sin^2 t\\ &= 1.\end{aligned}$$

Activity 5.1 Parametrising an ellipse

(a) Write down the standard parametrisation of the ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

(b) Calculate the points (x, y) corresponding to the parameter values $t = 0, \frac{1}{6}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi$ and $\frac{1}{2}\pi$. Plot these points, and hence sketch the ellipse, showing the direction of motion as t increases.

Solutions are given on page 54.

Parabola

The parabola in standard position has equation

$$y^2 = 4ax \quad (\text{where } a > 0).$$

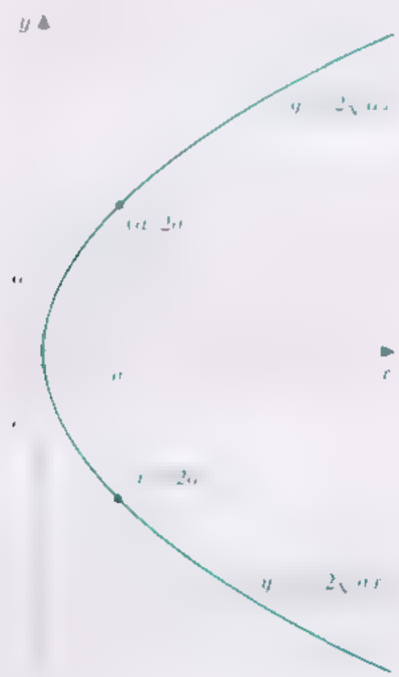


Figure 5.3 Parabola in standard position

For each value of $x > 0$, there are two values of y (namely, $y = \pm 2\sqrt{ax}$) for which (x, y) lies on the parabola: see Figure 5.3 On the other hand, for each real value of y, there is exactly one value of x, namely $x = y^2/(4a)$, for which (x, y) lies on the parabola. Therefore, we can use the y-coordinate as a parameter for this curve. Thus we set $y = t$, giving parametric equations

$$x = \frac{t^2}{4a}, \quad y = t.$$

In fact, a slightly different parametrisation is normally used for $y^2 = 4ax$. This uses $y = 2at$, rather than $y = t$, so

$$x = \frac{y^2}{4a} = \frac{(2at)^2}{4a} = at^2.$$

When no range is given for the parameter t, it is to be assumed that t can take every real value.

This **standard parametrisation of the parabola**,

$$x = at^2, \quad y = 2at,$$

has several advantages. For example, there are no fractions and also the three simple parameter values $t = -1$, 0 and 1 correspond to simple points on the parabola; see Figure 5.4, which shows the points in the following table.

t	-2	-1	0	1	2
$x(=at^2)$	$4a$	a	0	a	$4a$
$y(=2at)$	$-4a$	$-2a$	0	$2a$	$4a$

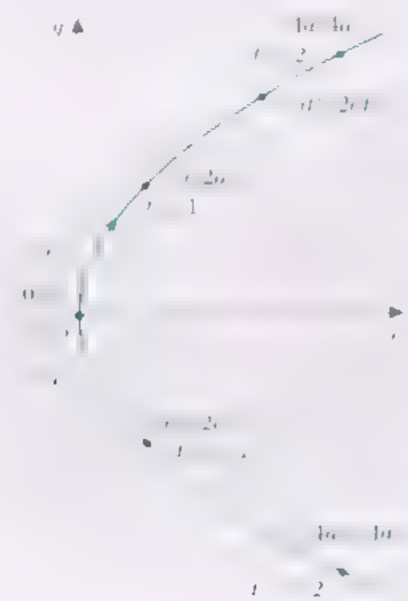


Figure 5.4 Standard parametrisation of the parabola

Notice that positive values of t give points (x, y) on the upper arm of the parabola, whereas negative values of t give points on the lower arm.

Activity 5.2 Parametrising a parabola

(a) Write down the standard parametrisation of the parabola $y^2 = 2x$.

(b) Calculate the points (x, y) corresponding to the parameter values $t = -1, 0, 1, 2$ and 3. Plot these points, and hence sketch the parabola, showing the direction of motion as t increases.

Solutions are given on page 54.

Hyperbola

Since the equation of the hyperbola in standard position,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{(where } a, b > 0\text{)}, \tag{5.2}$$

closely resembles that of the ellipse in standard position, it is natural to expect similar parametric equations. Such parametric equations can be obtained by using the trigonometric identity

$$\sec^2 t - \tan^2 t = 1, \tag{5.3}$$

where $\sec t = 1/\cos t$. This identity follows from the identity $\cos^2 t + \sin^2 t = 1$ on dividing by $\cos^2 t$ and then rearranging:

$$\frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t},$$

so

$$1 + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t};$$

that is,

$$1 + \tan^2 t = \frac{1}{\cos^2 t} = \sec^2 t,$$

as required. On comparing equations (5.2) and (5.3), we see that if

$$x = a \sec t \quad \text{and} \quad y = b \tan t, \tag{5.4}$$

then the point (x, y) lies on the hyperbola.

But how does the point (x, y) move along this hyperbola as the parameter t varies? For simplicity, we take $a = b = 1$ and consider some particular values of t.

t	0	[illegible]	[illegible]	[illegible]
$x(=\sec t)$	1	1.15	1.41	2
$y(=\tan t)$	0	0.58	1	1.73

For example,

$$\sec\left(\tfrac{1}{6}\pi\right) = \frac{1}{\cos\left(\frac{1}{6}\pi\right)} = \frac{1}{\sqrt{3}/2} \sim 1.15.$$

Figure 5.5 Points on a hyperbola

The four points in the table lie on the right-hand branch of the hyperbola, as indicated in Figure 5.5. You are now asked to calculate some further points on the hyperbola.

Activity 5.3 Points on a hyperbola

Use equation (5.4) with $a = 1$ and $b = 1$ to calculate points (x, y) corresponding to the parameter values $t = -\frac{1}{4}\pi, \frac{3}{4}\pi, \pi, \frac{5}{4}\pi$. Plot these points on a sketch of the hyperbola, showing the direction of motion on each branch as t increases.

A solution is given on page 55.

In Activity 5.3, you were not asked to calculate the points (x, y) corresponding to the parameter values $t = -\frac{1}{2}\pi$, $t = \frac{1}{2}\pi$ and $t = \frac{3}{2}\pi$. This is because, for these values of t, $\cos t = 0$, so $\sec t = 1/\cos t$ and $\tan t = \sin t/\cos t$ are not defined. Whenever t is near one of these values, $\cos t$ is close to 0, so $\sec t$ and $\tan t$ are both large in magnitude and the corresponding point $(x, y) = (\sec t, \tan t)$ is far from the origin.

It turns out that we obtain a parametrisation of the *whole* hyperbola given by equation (5.2) if we use equation (5.4) and let t range first from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ and then from $\frac{1}{2}\pi$ to $\frac{3}{2}\pi$ (excluding the points $-\frac{1}{2}\pi$, $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$ themselves). This gives the **standard parametrisation of the hyperbola**

$$x = a \sec t, \quad y = b \tan t \quad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi).$$

As t increases from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, the point $(a \sec t, b \tan t)$ moves upwards along the entire right-hand branch of the hyperbola. The left-hand branch is obtained as t increases from $\frac{1}{2}\pi$ to $\frac{3}{2}\pi$; see Figure 5.6.

Figure 5.6 shows one way of indicating that the parameter t runs from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, and from $\frac{1}{2}\pi$ to $\frac{3}{2}\pi$.

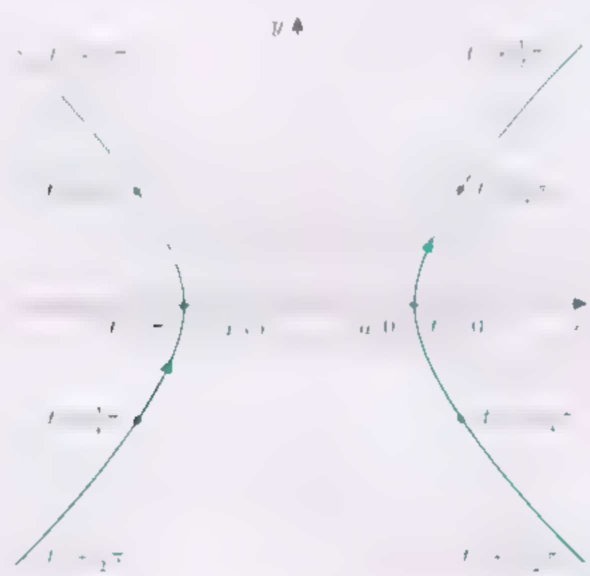


Figure 5.6 Standard parametrisation of the hyperbola

There are several alternative parametrisations of the hyperbola. For example, we can use the parametric equations (5.4) but with a different range of values of t such as, $0 \le t \le 2\pi$, excluding $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$. Alternatively, a parametrisation of just the right-hand branch is given by the equations

$$x = a\sqrt{t^2 + 1}, \quad y = bt.$$

5.2 Parametric equations for translated conics

The circle $(x-p)^2+(y-q)^2=r^2$ is congruent to the circle $x^2+y^2=r^2$, but has been translated so that its centre is at the point (p,q); see Figure 5.7 . Therefore, parametric equations for this translated circle are:

$$x=p+r\cos t,\quad y=q+r\sin t\quad (0\le t\le 2\pi).$$

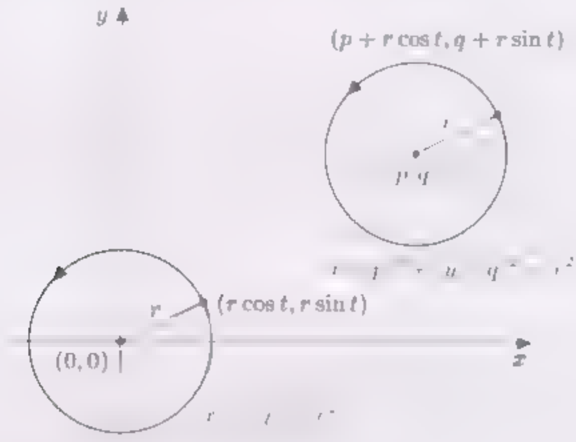


Figure 5.7 Parametrisation of a translated circle

Similarly, we can specify parametric equations for a translated conic. For example, in Section 4, Activity 4.2(a), you saw that the curve

$$3x^2+4y^2-48x-48y+324=0 \tag{5.5}$$

is congruent to the ellipse in standard position

$$\frac{x^2}{4}+\frac{y^2}{3}=1, \tag{5.6}$$

but has been translated so that its centre is the point $(8,6)$; see Figure 5.8. Since $a=2$ and $b=\sqrt{3}$ in equation (5.6), this ellipse has parametric equations

$$x=2\cos t,\quad y=\sqrt{3}\sin t\quad (0\le t\le 2\pi).$$

Therefore, the ellipse given by equation (5.5) has parametric equations

$$x=8+2\cos t,\quad y=6+\sqrt{3}\sin t\quad (0\le t\le 2\pi).$$

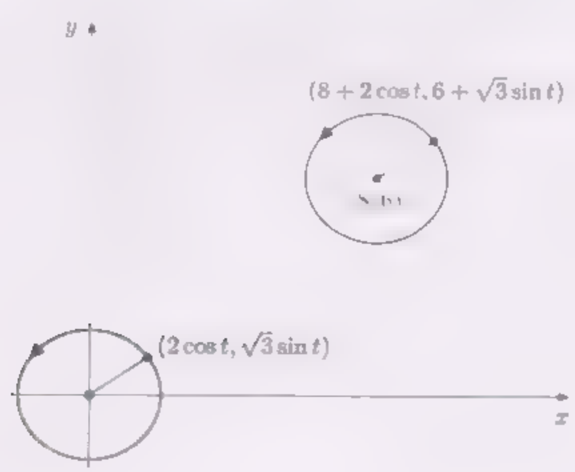


Figure 5.8 Parametrisation of a translated ellipse

Parametric equations for translated hyperbolas and parabolas can be obtained in a similar manner.

Activity 5.4 Parametrising translated conics

Write down parametric equations for each of the following quadratic curves. (You saw that these curves are translated conics in Section 4, Frame 3 and Activity 4.2(b).)

(a) $x^2 - 4y^2 - 6x - 40y - 95 = 0$

(b) $y^2 - 16x - 8y = 0$

Solutions are given on page 55.

Summary of Section 5

This section has introduced:

- the standard parametrisation of the ellipse

  $$x = a\cos t, \quad y = b\sin t \quad (0 \le t \le 2\pi);$$

- the standard parametrisation of the parabola

  $$x = at^2, \quad y = 2at;$$

- the standard parametrisation of the hyperbola

  $$x = a\sec t, \quad y = b\tan t \quad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi,\ \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi);$$

- parametrisations for translated conics.

Exercises for Section 5

Exercise 5.1

Classify each of the following curves as ellipse, parabola or hyperbola and then write down a parametrisation for each curve.

(a) $x^2 + 2y^2 = 4$ (b) $x^2 - 9 - 9y^2 = 0$ (c) $6x - 2y^2 = 0$

Exercise 5.2

Use the solution to Exercise 4.1 to write down parametric equations for each of the following quadratic curves.

(a) $3x^2 - 2y^2 + 6x + 12y - 33 = 0$

(b) $x^2 + 16y^2 - 10x + 64y + 73 = 0$

6 Conics on the computer

The work in this section is to use the computer to plot conics using parametric equations.

Refer to Computer Book A for the work in this section.

Summary of Section 6

This section has introduced:

- techniques for plotting conics in standard position and indicating a particular point on a conic;
- techniques for plotting translated conics.

Summary of Chapter A2

In this chapter, you met a family of curves, the conics, which have an ancient geometric definition. Conics have become of great importance because of their many occurrences and uses, and because they can be handled effectively using algebra.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Double cone, empty set, vertex of a cone, ellipse, parabola, hyperbola, degenerate conic, non-degenerate conic, ellipse/hyperbola/parabola in standard position, centre of ellipse and hyperbola, vertex of conic, major and minor axes of ellipse, asymptote of hyperbola, focus, directrix, eccentricity, quadratic curve, translated conic, standard parametrisation.

Symbols and notation to know and use

$\simeq$, Pd, e (eccentricity)

Mathematical skills

- ◇ Sketch a conic in standard position and a translated conic.
- ◇ Relate the algebraic and focus–directrix definitions of a conic.
- ◇ Find the focus (foci), directrix (directrices) and eccentricity of a conic in standard position.
- ◇ Classify a quadratic curve with equation of the form

  $$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

  as an ellipse, parabola or hyperbola, where possible.
- ◇ Specify the standard parametrisation of a given ellipse, parabola or hyperbola in standard position.
- ◇ Specify a parametrisation of a given translated ellipse, parabola or hyperbola.

Mathcad skills

- ◇ Plot a conic in standard position, or a translated conic, using parametrisation.

Ideas to be aware of

- ◇ That mathematical objects, such as conics, may have different representations (algebraic or geometric), each with their own advantages and disadvantages.
- ◇ The idea that a curve may belong to a family of curves, all of which have similar properties.

Solutions to Activities

Solution 1.1

(a) If we slice the (horizontal) cylinder with a vertical plane, then the cross-section is a circle.

(b) If we slice the cylinder with a tilted plane, as shown in the figure, then the cross-section is an oval. The further we tilt the plane, the more 'elongated' is the oval.

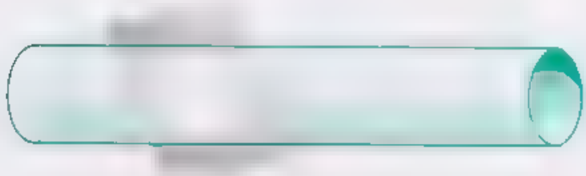


Figure S.1

(c) If we slice the cylinder with a horizontal plane, then the cross-section is one of the following:

- a pair of parallel lines;
- a single line (this occurs when the plane just touches the cylinder);
- the empty set (this occurs when the plane does not meet the cylinder).

Solution 2.1

(a) (i) On dividing $9x^2 + 16y^2 = 144$ throughout by 144 (to give 1 on the right-hand side), we obtain

$$\frac{x^2}{16} + \frac{y^2}{9} = 1; \quad \text{that is,} \quad \frac{x^2}{4^2} + \frac{y^2}{3^2} = 1.$$

This is in the form of equation (2.1), with $a = 4$ and $b = 3$. The graph of the ellipse is as follows.

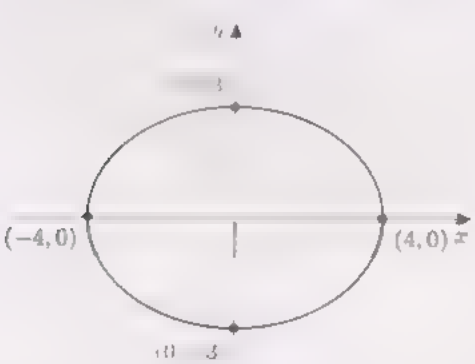


Figure S.2

(ii) First rearrange the equation so that the constant term appears on the right-hand side of the equation. On dividing $x^2 + 4y^2 = 4$ throughout by 4 (to give 1 on the right-hand side), we obtain

$$\frac{x^2}{4} + y^2 = 1; \quad \text{that is,} \quad \frac{x^2}{2^2} + \frac{y^2}{1^2} = 1.$$

This is in the form of equation (2.1), with $a = 2$ and $b = 1$. The graph of the ellipse is as follows.

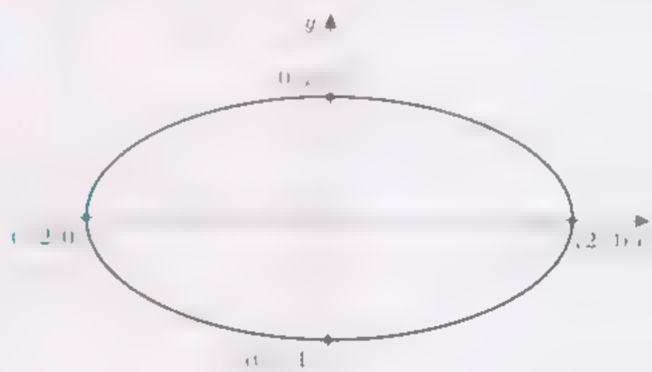


Figure S.3

(b) On dividing $16x^2 + 9y^2 = 144$ throughout by 144 (to give 1 on the right-hand side), we obtain

$$\frac{x^2}{9} + \frac{y^2}{16} = 1; \quad \text{that is,} \quad \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1.$$

This is in the form of equation (2.1) with $a = 3$ and $b = 4$ (so that $a < b$). This equation is the same as that in part(a)(i) but with the roles of x and y interchanged. The graph of the ellipse is as follows. (This graph may be obtained by reflecting the ellipse in part (a)(i) in the line $y = x$.)

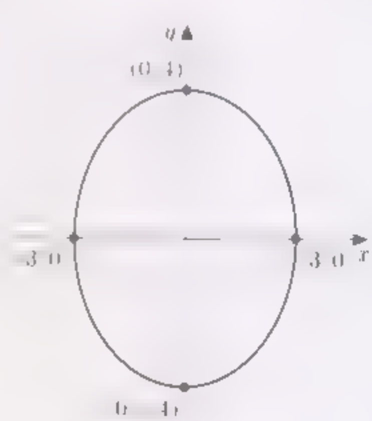


Figure S.4

Solution 2.2

(a) On dividing $9x^2 - 16y^2 = 144$ throughout by 144 (to give 1 on the right-hand side), we obtain

$$\frac{x^2}{16} - \frac{y^2}{9} = 1; \quad \text{that is,} \quad \frac{x^2}{4^2} - \frac{y^2}{3^2} = 1.$$

This is in the form of equation (2.4), with $a = 4$ and $b = 3$. The asymptotes are the lines $y = \pm\frac{3}{4}x$. The graph of the hyperbola is as follows.

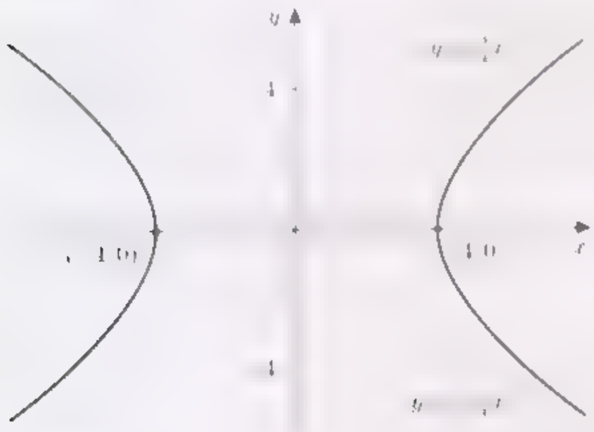


Figure S.5

(b) Here the constant 1 already appears on the right-hand side. On rewriting the coefficients on the left-hand side as $4 = 1/(1/4)$ and $3 = 1/(1/3)$, we obtain

$$\frac{x^2}{1/4} - \frac{y^2}{1/3} = 1; \quad \text{that is,} \quad \frac{x^2}{(1/2)^2} - \frac{y^2}{(1/\sqrt{3})^2} = 1$$

This is in the form of equation (2.4), with $a = 1/2$ and $b = 1/\sqrt{3} \simeq 0.58$. The asymptotes are the lines $y = \pm(2/\sqrt{3})x$. Since $2/\sqrt{3} \simeq 1.15$, the graph of the hyperbola is as follows.

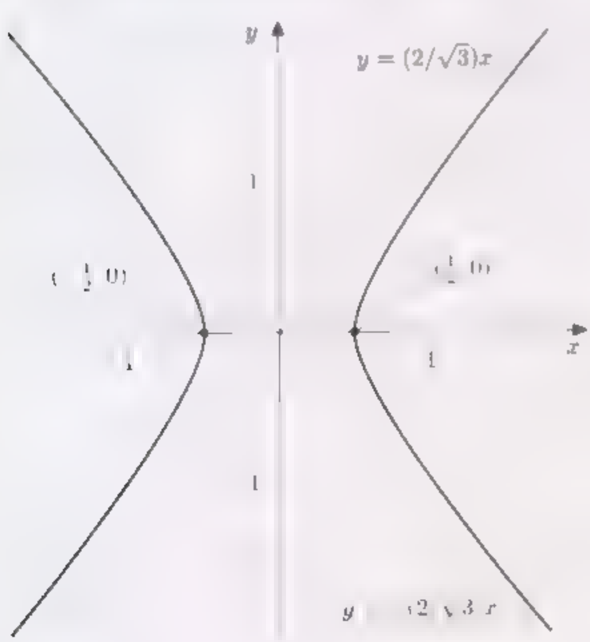


Figure S.6

Solution 2.3

(a) To write the equation $y^2 = x$ in the form of equation (2.7), we replace x by $(4 \times \frac{1}{4})x$. Now

$$y^2 = (4 \times \tfrac{1}{4})x$$

is in the form of equation (2.7) with $a = \frac{1}{4}$. The graph of the parabola is as follows.

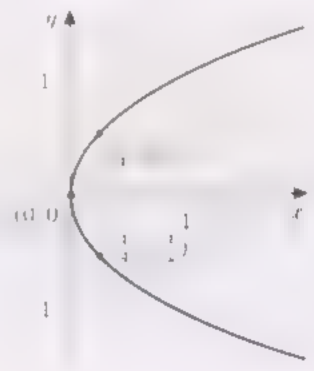


Figure S.7

(b) To write the equation $4y^2 - 2x = 0$ in the form of equation (2.7), we first rearrange the equation to isolate y^2 on the left-hand side and obtain $y^2 = \frac{1}{2}x$. Then we replace $\frac{1}{2}x$ by $(4 \times \frac{1}{8})x$. Now

$$y^2 = (4 \times \tfrac{1}{8})x$$

is in the form of equation (2.7) with $a = \frac{1}{8}$. The graph of the parabola is as follows.

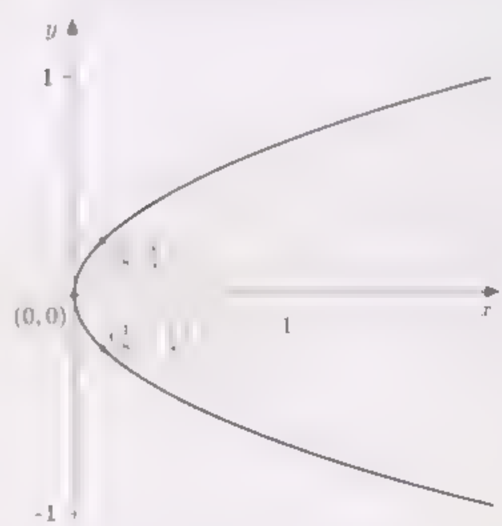


Figure S.8

Solution 3.1

(a) The distance from (x, y) to the point $(a, 0)$ is

$$\sqrt{(x-a)^2+y^2} = \sqrt{x^2-2ax+a^2+y^2}.$$

(b) The distance from (x, y) to the line $x = -a$ is

$$x + a$$

since $x \geq 0$ for all points on the parabola.

(c) Using the fact that $y^2 = 4ax$ for a point on the parabola, we obtain

$$\begin{aligned}\sqrt{x^2-2ax+a^2+y^2} &= \sqrt{x^2-2ax+a^2+4ax}\\ &= \sqrt{x^2+2ax+a^2}\\ &= \sqrt{(x+a)^2}\\ &= x+a,\end{aligned}$$

since $x + a \geq 0$ when (x, y) lies on the parabola. Thus the distances in parts (a) and (b) are the same.

Solution 3.2

(a) The equation $y^2 = x$ is of the form $y^2 = (4 \times \frac{1}{4})x$, so $a = \frac{1}{4}$ in this case. Thus the focus is the point $(\frac{1}{4}, 0)$ and the directrix is the line $x = -\frac{1}{4}$, as shown in the figure.

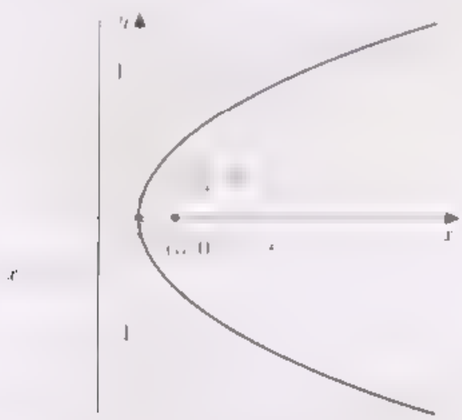


Figure S.9

(b) The equation $4y^2 - 2x = 0$ is of the form $y^2 = (4 \times \frac{1}{8})x$, so $a = \frac{1}{8}$ in this case. Thus the focus is the point $(\frac{1}{8}, 0)$ and the directrix is the line $x = -\frac{1}{8}$, as shown in the figure.

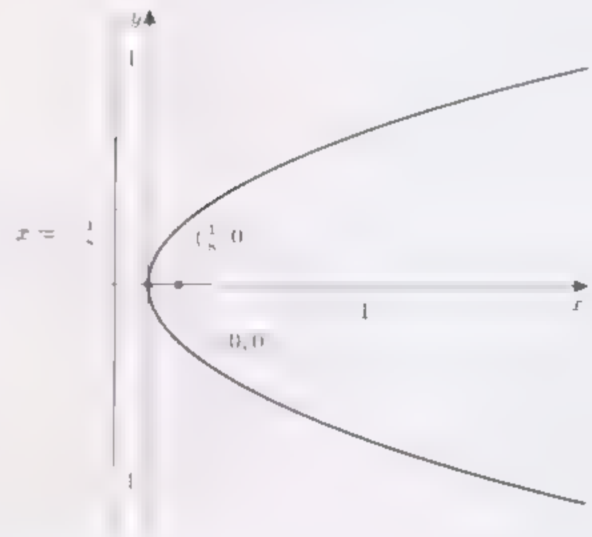


Figure S.10

Solution 3.3

(a)
$$\begin{aligned}PF &= \sqrt{(x-ae)^2+y^2}\\ &= \sqrt{x^2-2aex+a^2e^2+y^2}.\end{aligned}$$

Since any point P which satisfies $PF = ePd$ must lie to the left of the line $x = a/e$, we have

$$Pd = a/e - x.$$

(b) Substituting the expressions from part (a) into $PF = ePd$, we obtain the equation

$$\begin{aligned}\sqrt{x^2-2aex+a^2e^2+y^2} &= e(a/e-x)\\ &= a-ex.\end{aligned}$$

(c) To remove the square root from the equation in part (b), we square both sides:

$$\begin{aligned}x^2-2aex+a^2e^2+y^2 &= (a-ex)^2\\ &= a^2-2aex+e^2x^2.\end{aligned}$$

By cancelling and collecting like terms, we obtain

$$x^2(1-e^2)+y^2 = a^2(1-e^2),$$

and on dividing throughout by $a^2(1-e^2)$, we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)} = 1.$$

This is the equation of an ellipse in standard position where $b^2 = a^2(1-e^2)$; that is, $b = a\sqrt{1-e^2}$.

(d) Since $b^2 = a^2(1-e^2)$, we have

$$\frac{b^2}{a^2} = 1-e^2; \quad \text{that is,} \quad e^2 = 1-\frac{b^2}{a^2}.$$

So $e = \sqrt{1-b^2/a^2}$, since $e > 0$.

Solution 3.4

Note that in each case we find the eccentricity first, since the formulas for the foci and directrices both involve e.

(a) The equation $9x^2 + 16y^2 = 144$ can be rearranged as $x^2/4^2 + y^2/3^2 = 1$, so $a = 4$ and $b = 3$. The eccentricity of this ellipse is

$$e = \sqrt{1 - b^2/a^2} = \sqrt{1 - 9/16} = \sqrt{7}/4.$$

Thus

$$ae = 4 \times \sqrt{7}/4 = \sqrt{7} \simeq 2.65,$$

so the foci of this ellipse are (approximately) the points $(2.65, 0)$ and $(-2.65, 0)$. Also

$$a/e = 4/(\sqrt{7}/4) = 16/\sqrt{7} \simeq 6.05,$$

so the corresponding directrices of this ellipse are the lines $x = 6.05$ and $x = -6.05$.

The foci and directrices are shown in the figure.

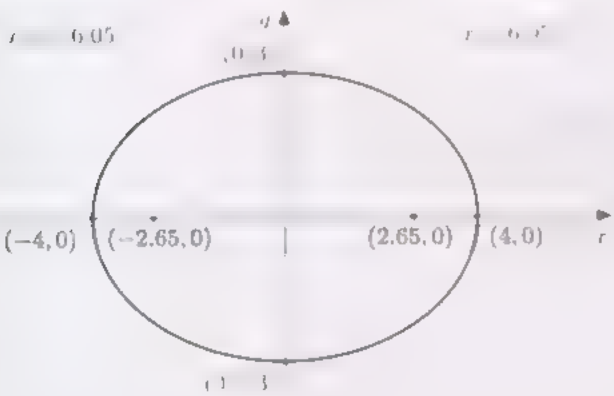


Figure S.11

(When sketching points and curves, it is often convenient to use decimal approximations rather than exact values.)

(b) The equation $x^2 + 4y^2 - 4 = 0$ can be rearranged as $x^2/2^2 + y^2/1^2 = 1$, so $a = 2$ and $b = 1$. The eccentricity of this ellipse is

$$e = \sqrt{1 - b^2/a^2} = \sqrt{1 - 1/4} = \sqrt{3}/2.$$

Thus

$$ae = 2 \times \sqrt{3}/2 = \sqrt{3} \simeq 1.73,$$

so the foci of this ellipse are the points $(1.73, 0)$ and $(-1.73, 0)$. Also

$$a/e = 2/(\sqrt{3}/2) = 4/\sqrt{3} \simeq 2.31,$$

so the corresponding directrices of this ellipse are the lines $x = 2.31$ and $x = -2.31$.

The foci and directrices are shown in the figure.

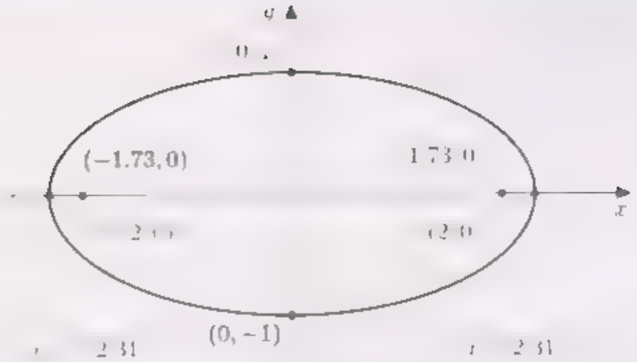


Figure S.12

Solution 3.5

First notice that the point $P(x, y)$ can lie on either side of the line $x = 1$. Consider each case separately.

Suppose first that the point $P(x, y)$ lies to the right of the line $x = 1$. Then $Pd = x - 1$ and $PF = \sqrt{(x-2)^2 + y^2}$. Using the equation $PF = \sqrt{2}Pd$, we obtain

$$\sqrt{(x-2)^2 + y^2} = \sqrt{2}(x - 1)$$

and then

$$\sqrt{x^2 - 4x + 4 + y^2} = \sqrt{2}(x - 1).$$

To remove the square roots from this equation, we square both sides:

$$\begin{aligned} x^2 - 4x + 4 + y^2 &= 2(x-1)^2 \\ &= 2(x^2 - 2x + 1) \\ &= 2x^2 - 4x + 2. \end{aligned}$$

By cancelling and collecting like terms, we obtain

$$x^2 - y^2 = 2,$$

and on dividing throughout by 2, we obtain

$$\frac{x^2}{2} - \frac{y^2}{2} = 1$$

as required.

Now suppose that the point $P(x, y)$ lies to the left of the line $x = 1$. Then $Pd = 1 - x$ and $PF = \sqrt{(x-2)^2 + y^2}$. Using the equation $PF = \sqrt{2}Pd$, we obtain

$$\sqrt{(x-2)^2 + y^2} = \sqrt{2}(1 - x)$$

and then

$$\sqrt{x^2 - 4x + 4 + y^2} = \sqrt{2}(1 - x).$$

To remove the square root from this equation, we square both sides:

$$\begin{aligned} x^2 - 4x + 4 + y^2 &= 2(1 - x)^2 \\ &= 2(1 - 2x + x^2) \\ &= 2 - 4x + 2x^2. \end{aligned}$$

This equation is just a rearrangement of the equation in the first case so, by cancelling, collecting like terms, and dividing throughout by 2, we obtain the same equation

$$\frac{x^2}{2} - \frac{y^2}{2} = 1$$

Solution 3.6

(a) The equation $9x^2 - 16y^2 = 144$ can be rearranged as $x^2/4^2 - y^2/3^2 = 1$, so $a = 4$ and $b = 3$. The eccentricity of this hyperbola is

$$e = \sqrt{1 + b^2/a^2} = \sqrt{1 + 9/16} = 5/4.$$

Thus

$$ae = 4 \times 5/4 = 5,$$

so the foci of this hyperbola are the points $(5, 0)$ and $(-5, 0)$. Also

$$a/e = 4/(5/4) = 16/5 = 3.2,$$

so the corresponding directrices of this hyperbola are the lines $x = 3.2$ and $x = -3.2$.

The foci and directrices are shown in the figure.

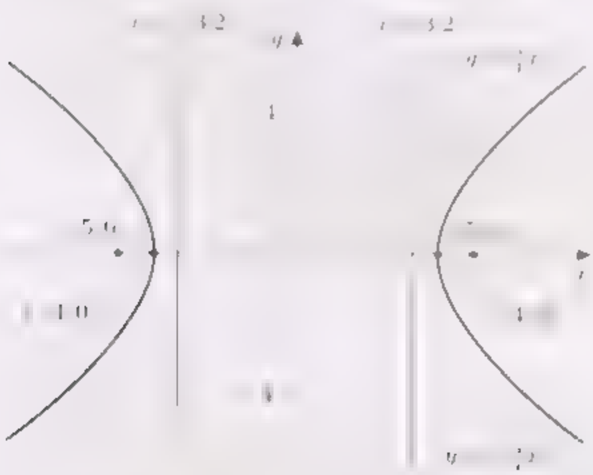


Figure S.13

(b) The equation $4x^2 - 3y^2 = 1$ can be rearranged as $x^2/(1/2)^2 - y^2/(1/\sqrt{3})^2 = 1$, so $a = 1/2$ and $b = 1/\sqrt{3}$. The eccentricity of this hyperbola is

$$e = \sqrt{1 + b^2/a^2} = \sqrt{1 + \frac{4}{3}} = \sqrt{\frac{7}{3}}.$$

Thus

$$ae = \frac{1}{2} \times \sqrt{\frac{7}{3}} = \frac{\sqrt{7}}{2\sqrt{3}} \simeq 0.76,$$

so the foci of this hyperbola are the points $(0.76, 0)$ and $(-0.76, 0)$. Also

$$a/e = \frac{1/2}{\sqrt{7/3}} = \frac{\sqrt{3}}{2\sqrt{7}} \simeq 0.33,$$

so the corresponding directrices of this hyperbola are the lines $x = 0.33$ and $x = -0.33$.

The foci and directrices are shown in the figure.

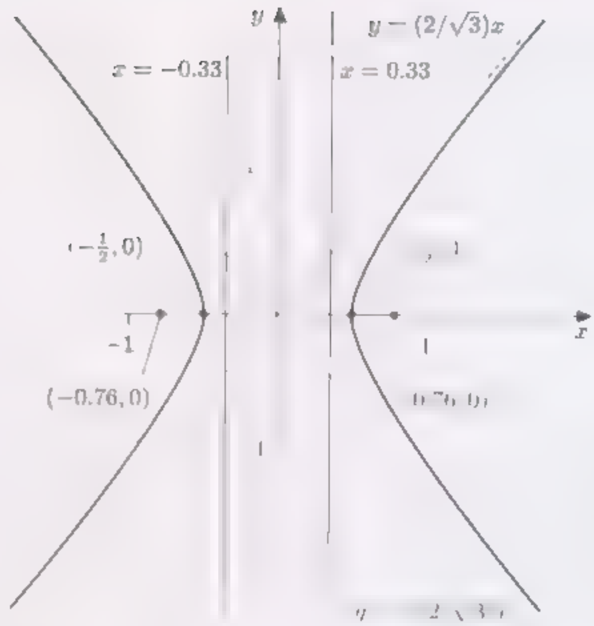


Figure S.14

Solution 4.1

(a) $A = 4$, $B = 0$, $C = -9$, $D = 0$, $E = 0$, $F = -36$.

(b) $A = 0$, $B = 0$, $C = 1$, $D = -4$, $E = 0$, $F = 0$.

(c) $A = 0$, $B = 1$, $C = 0$, $D = 0$, $E = 0$, $F = -1$.

Solution 4.2

(a) On collecting x and y terms, we obtain

$$3(x^2 - 16x) + 4(y^2 - 12y) + 324 = 0.$$

On completing the squares, we then obtain

$$3((x - 8)^2 - 64) + 4((y - 6)^2 - 36) + 324 = 0;$$

that is,

$$3(x - 8)^2 + 4(y - 6)^2 = 12,$$

or

$$\frac{(x - 8)^2}{4} + \frac{(y - 6)^2}{3} = 1.$$

This is the ellipse in standard position with $a = 2$, $b = \sqrt{3}$, translated to have centre $(8, 6)$.

The ellipse is shown overleaf.

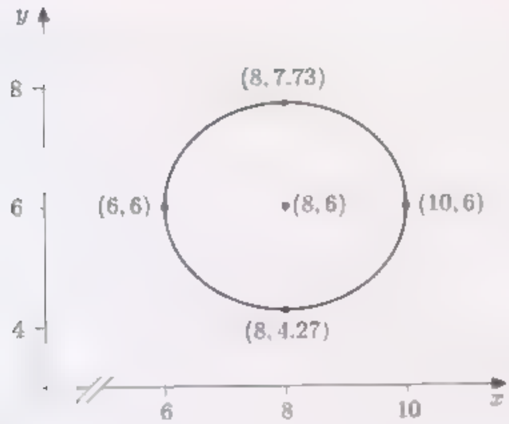


Figure S.15

(b) On collecting y terms, we obtain

$$(y^2 - 8y) - 16x = 0.$$

On completing the square, we then obtain

$$(y - 4)^2 - 16 - 16x = 0;$$

that is,

$$(y - 4)^2 = 16(x + 1) = (4 \times 4)(x + 1).$$

This is the parabola in standard position with $a = 4$, translated to have vertex $(-1, 4)$.

The parabola is as follows.

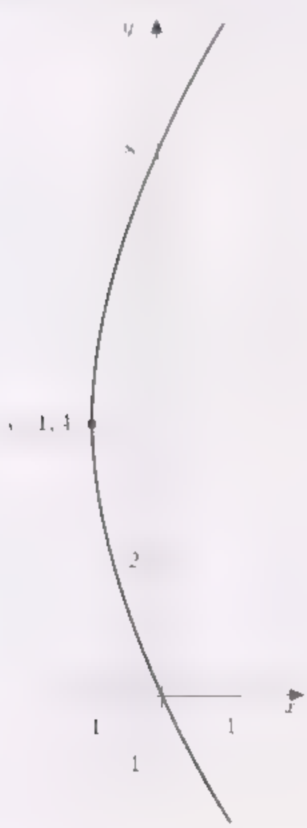


Figure S.16

Solution 5.1

(a) Since $a = 3$ and $b = 2$, the standard parametrisation is

$$x = 3\cos t, \quad y = 2\sin t \quad (0 \le t < 2\pi).$$

(b) The values of x and y for the given values of t are in the following table, accurate to two decimal places.

t	0	[illegible]	[illegible]	[illegible]	[illegible]
x	3	2.6	2.12	1.5	0
y	0	1	1.41	1.73	2

Thus the points $(3, 0)$, $(2.6, 1)$, $(2.12, 1.41)$, $(1.5, 1.73)$ and $(0, 2)$ lie on this ellipse.

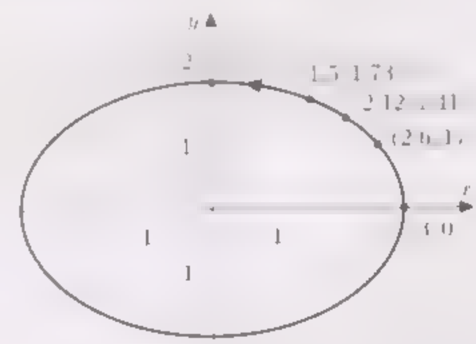


Figure S.17

Solution 5.2

(a) Since $a = \frac{1}{2}$, the standard parametrisation is

$$x = \tfrac{1}{2}t^2, \quad y = t.$$

(b) The values of x and y for the given values of t are in the following table.

t	−1	0	1	2	3
x	0.5	0	0.5	2	4.5
y	−1	0	1	2	3

Thus the points $(0.5, -1)$, $(0, 0)$, $(0.5, 1)$, $(2, 2)$ and $(4.5, 3)$ lie on this parabola.

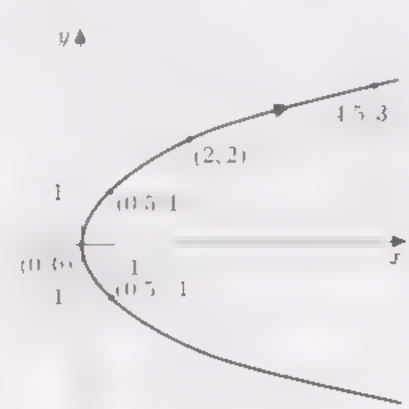


Figure S.18

Solution 5.3

The values of x and y for the given values of t are in the following table, accurate to two decimal places.

t	$-\frac{1}{4}\pi$	$\frac{3}{4}\pi$	π	[illegible]
x	1.41	-1.41	[illegible]	[illegible]
y	[illegible]	[illegible]	[illegible]	[illegible]

These values can be found by considering the coordinates of the points on the unit circle with angles $\frac{1}{4}\pi$, $\frac{3}{4}\pi$, π and $\frac{5}{4}\pi$. For example, the point on the unit circle with angle $-\frac{1}{4}\pi$ has coordinates $(1/\sqrt{2}, -1/\sqrt{2})$. Hence

$$x = \sec(-\tfrac{1}{4}\pi) = \frac{1}{\cos(-\frac{1}{4}\pi)}$$

$$= \frac{1}{[illegible]}$$

$$\frac{1}{[illegible]} = 1.41$$

Thus the points $(1.41, -1)$, $(-1.41, -1)$, $(-1, 0)$ and $(-1.41, 1)$ lie on the hyperbola.

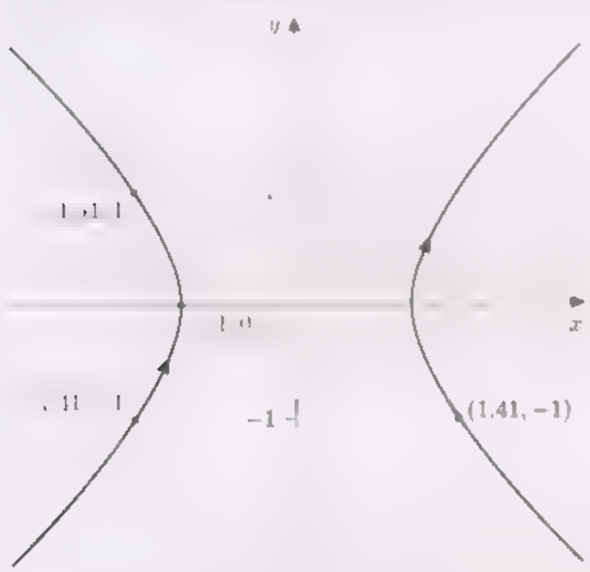


Figure S.19

Solution 5.4

(a) As you saw in Frame 3,

$$x^2 - 4y^2 - 6x - 40y - 95 = 0$$

can be rearranged as

$$\frac{(x-3)^2}{4} - (y+5)^2 = 1.$$

Thus the curve is the hyperbola in standard position with $a = 2$ and $b = 1$, translated to have centre $(3, -5)$. A parametrisation of this curve is, therefore,

$$x = 3 + 2\sec t, \quad y = -5 + \tan t$$
$$(-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi).$$

(b) As you saw in the solution to Activity 4.2(b), the equation

$$y^2 - 16x - 8y = 0$$

can be rearranged as

$$(y-4)^2 = 16(x+1).$$

Thus the curve is the parabola in standard position with $a = 4$, translated to have vertex $(-1, 4)$. A parametrisation of this curve is, therefore,

$$x = -1 + 4t^2, \quad y = 4 + 8t.$$

Solutions to Exercises

Solution 2.1

(a) On dividing $3x^2 + 4y^2 = 192$ throughout by 192 (to give 1 on the right-hand side), we obtain

$$\frac{x^2}{64} + \frac{y^2}{48} = 1; \quad \text{that is,} \quad \frac{x^2}{8^2} + \frac{y^2}{[illegible]} = 1$$

This is in the form of equation (2.1), with $a = 8$ and $b = \sqrt{48} = 4\sqrt{3} \simeq 6.93$. The graph of the ellipse is as follows.

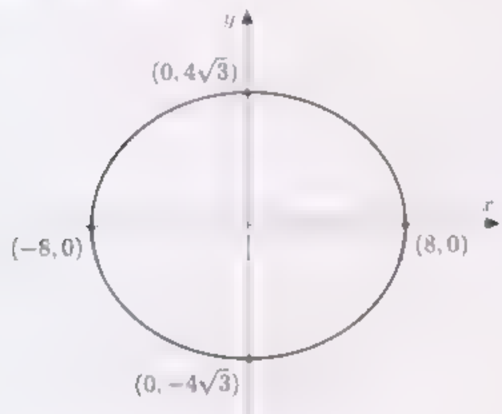


Figure S.20

(b) First rearrange the equation so that the constant term appears on the right-hand side of the equation. On dividing $x^2 + 18y^2 = 9$ throughout by 9 (to give 1 on the right-hand side), we obtain

$$\frac{x^2}{9} + 2y^2 = 1; \quad \text{that is,} \quad \frac{x^2}{3^2} + \frac{y^2}{(1/\sqrt{2})^2} = 1.$$

This is in the form of equation (2.1), with $a = 3$ and $b = 1/\sqrt{2} \simeq 0.71$. The graph of the ellipse is as follows.

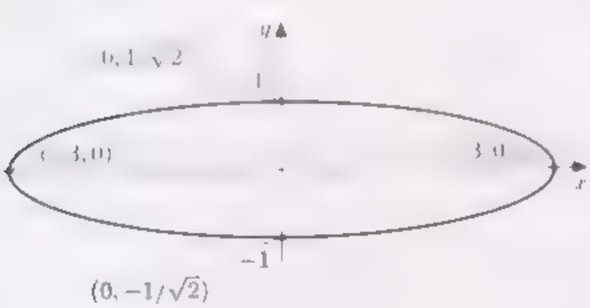


Figure S.21

Solution 2.2

(a) First rearrange the equation so that the constant term appears on the right-hand side of the equation. On dividing $x^2 - 9y^2 = 9$ throughout by 9 (to give 1 on the right-hand side), we obtain

$$\frac{x^2}{9} - y^2 = 1; \quad \text{that is,} \quad \frac{x^2}{3^2} - \frac{y^2}{1^2} = 1.$$

This is in the form of equation (2.4), with $a = 3$ and $b = 1$. The asymptotes are the lines $y = \pm\frac{1}{3}x$. The graph of the hyperbola is as follows.

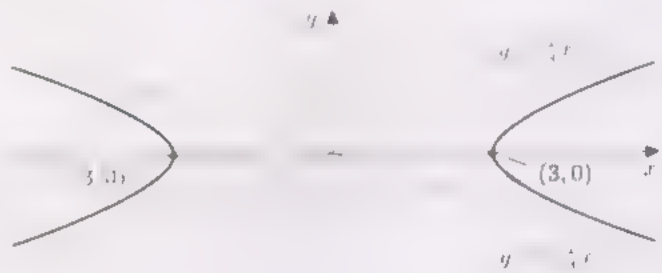


Figure S.22

(b) First rearrange the equation so that 1 appears on the right-hand side of the equation: $25x^2 - 4y^2 = 1$. On rewriting the coefficients on the left-hand side as $25 = 1/(1/25)$ and $4 = 1/(1/4)$, we obtain

$$\frac{x^2}{1/25} - \frac{y^2}{1/4} = 1$$

that is,

$$\frac{x^2}{(1/5)^2} - \frac{y^2}{(1/2)^2} = 1$$

This is in the form of equation (2.4), with $a = \frac{1}{5}$ and b [illegible]. The asymptotes are the lines y [illegible]. The graph of the hyperbola is as follows.

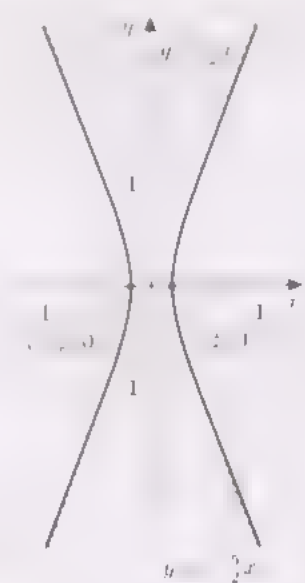


Figure S.23

Solution 2.3

To write the equation $6x - 2y^2 = 0$ in the form of equation (2.7), we first rearrange the equation to isolate y^2 and obtain $y^2 = 3x$. Then we replace $3x$ by $(4 \times \frac{3}{4})x$. Now

$$y^2 = (4 \times \tfrac{3}{4})x$$

is in the form of equation (2.7) with $a = \frac{3}{4}$. The graph of the parabola is as follows.

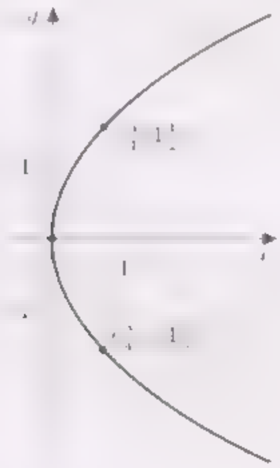


Figure S.24

Solution 3.1

The equation $6x - 2y^2 = 0$ can be rearranged as $y^2 = (4 \times \frac{3}{4})x$, so $a = \frac{3}{4}$ in this case. Thus the focus is the point $(\frac{3}{4}, 0)$ and the directrix is the line $x = -\frac{3}{4}$, as shown in the figure.

Figure S.25

Solution 3.2

(a) The equation $3x^2 + 4y^2 = 192$ can be rearranged as $x^2/8^2 + y^2/(\sqrt{48})^2 = 1$, so $a = 8$ and $b = 4\sqrt{3} \simeq 6.93$. The eccentricity of this ellipse is

$$e = \sqrt{1 - b^2/a^2} = \sqrt{1 - 48/64} = 1/2.$$

Thus

$$ae = 8 \times 1/2 = 4,$$

so the foci of this ellipse are the points $(4, 0)$ and $(-4, 0)$. Also

$$a/e = 8/(1/2) = 16,$$

so the corresponding directrices of this ellipse are the lines $x = 16$ and $x = -16$.

The foci and directrices are shown in the figure.

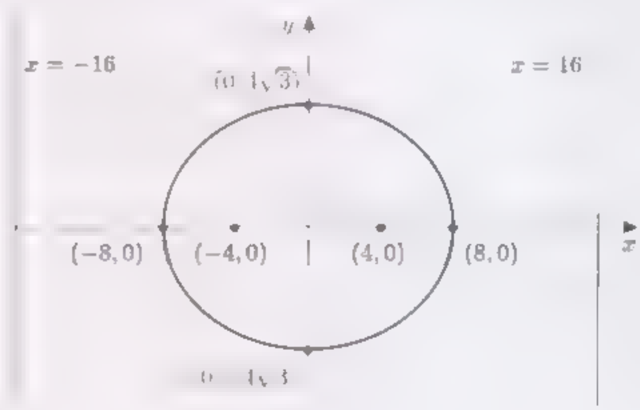


Figure S.26

(b) The equation $x^2 + 18y^2 - 9 = 0$ can be rearranged as $x^2/3^2 + y^2/(1/\sqrt{2})^2 = 1$, so $a = 3$ and $b = 1/\sqrt{2} \simeq 0.71$. The eccentricity of this ellipse is

$$e = \sqrt{1 - b^2/a^2} = \sqrt{1 - \frac{1/2}{9}} = \sqrt{\frac{17}{18}}.$$

Thus

$$ae = 3 \times \sqrt{\frac{17}{18}} = \sqrt{\frac{17}{2}} \simeq 2.92,$$

so the foci of this ellipse are the points $(2.92, 0)$ and $(-2.92, 0)$. Also

$$a/e = \frac{3}{\sqrt{17/18}} = 9\sqrt{\frac{2}{17}} \simeq 3.09,$$

so the corresponding directrices of this ellipse are the lines $x = 3.09$ and $x = -3.09$.

In this case the foci and directrices are so close to the vertices on the x-axis that the figure looks like Figure S.21 with vertical lines added through these vertices. It is omitted!

Solution 3.3

(a) The equation $x^2 - 9 - 9y^2 = 0$ can be rearranged as $x^2/3^2 - y^2/1^2 = 1$, so $a = 3$ and $b = 1$. The eccentricity of this hyperbola is

$$e = \sqrt{1 + b^2/a^2} = \sqrt{1 + 1/9} = \sqrt{10}/3.$$

Thus

$$ae = 3 \times \sqrt{10}/3 = \sqrt{10} \simeq 3.16,$$

so the foci of this hyperbola are the points $(3.16, 0)$ and $(-3.16, 0)$. Also

$$a/e = 3/(\sqrt{10}/3) = 9/\sqrt{10} \simeq 2.85,$$

so the corresponding directrices of this hyperbola are the lines $x = 2.85$ and $x = -2.85$.

(b) The equation $4y^2 - 25x^2 + 1 = 0$ can be rearranged as $x^2/(1/5)^2 - y^2/(1/2)^2 = 1$, so $a = \frac{1}{5}$ and $b = \frac{1}{2}$. The eccentricity of this hyperbola is

$$e = \sqrt{1 + b^2/a^2} = \sqrt{1 + 25/4} = \sqrt{29}/2.$$

Thus

$$ae = \frac{1}{5} \times \frac{\sqrt{29}}{2} = \frac{\sqrt{29}}{10} \simeq 0.54,$$

so the foci of this hyperbola are the points $(0.54, 0)$ and $(-0.54, 0)$. Also

$$a/e = \frac{1/5}{\sqrt{29}/2} = \frac{2}{5\sqrt{29}} \simeq 0.07,$$

so the corresponding directrices of this hyperbola are the lines $x = 0.07$ and $x = -0.07$.

Solution 4.1

(a) On collecting x and y terms, we obtain

$$3(x^2 + 2x) - 2(y^2 - 6y) = 33.$$

On completing the squares, we then obtain

$$3((x+1)^2 - 1) - 2((y-3)^2 - 9) = 33;$$

that is,

$$3(x+1)^2 - 2(y-3)^2 = 18,$$

or

$$\frac{(x+1)^2}{6} - \frac{(y-3)^2}{9} = 1.$$

This is the hyperbola in standard position with $a = \sqrt{6} \simeq 2.45$, $b = 3$, translated to have centre $(-1, 3)$. The hyperbola is as follows.

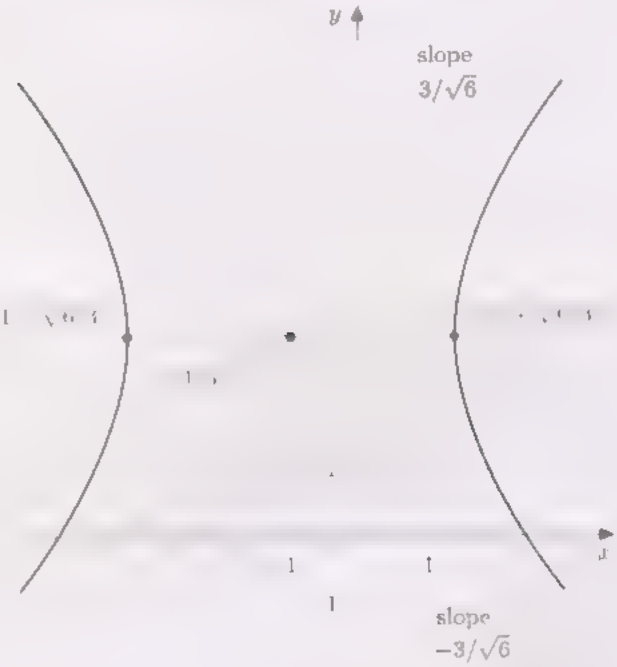


Figure S.27

(b) On collecting x and y terms, we obtain

$$(x^2 - 10x) + 16(y^2 + 4y) = -73.$$

On completing the squares, we then obtain

$$(x-5)^2 - 25 + 16((y+2)^2 - 4) = -73;$$

that is,

$$(x-5)^2 + 16(y+2)^2 = 16,$$

or

$$\frac{(x-5)^2}{16} + \frac{(y+2)^2}{1} = 1.$$

This is the ellipse in standard position with $a = 4$, $b = 1$, translated to have centre $(5, -2)$. The ellipse is as follows.

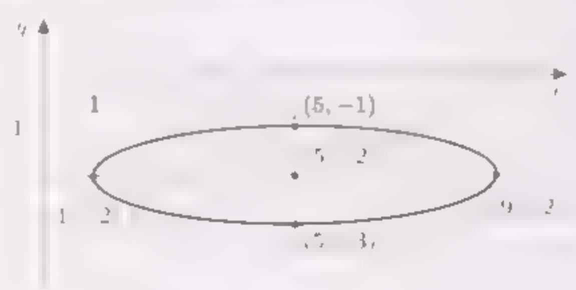


Figure S.28

Solution 5.1

(a) This equation can be rearranged as

$$\frac{x^2}{4} + \frac{y^2}{2} = 1,$$

so this curve is an ellipse in standard position with $a = 2$, $b = \sqrt{2}$. The standard parametrisation is

$$x = 2\cos t, \quad y = \sqrt{2}\sin t \quad (0 \le t \le 2\pi).$$

(b) This equation can be rearranged as

$$\frac{x^2}{9} - \frac{y^2}{1} = 1,$$

so this curve is a hyperbola in standard position with parameters $a = 3$, $b = 1$. The standard parametrisation is

$$x = 3\sec t, \quad y = \tan t$$
$$(-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi).$$

(c) This equation may be rearranged as

$$y^2 = (4 \times \tfrac{3}{4})x,$$

so this curve is a parabola in standard position with parameter $a = \frac{3}{4}$. The standard parametrisation is

$$x = \tfrac{3}{4}t^2, \quad y = \tfrac{3}{2}t.$$

Solution 5.2

(a) From the solution to Exercise 4.1(a), we know that the equation $3x^2 - 2y^2 + 6x + 12y - 33 = 0$ can be rearranged as

$$\frac{(x+1)^2}{6} - \frac{(y-3)^2}{9} = 1.$$

This is the hyperbola in standard position with $a = \sqrt{6}$, $b = 3$, translated to have centre $(-1, 3)$, so a parametrisation is

$$x = -1 + \sqrt{6}\sec t, \quad y = 3 + 3\tan t$$
$$(-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi).$$

(b) From the solution to Exercise 4.1(b), we know that the equation $x^2 + 16y^2 - 10x + 64y + 73 = 0$ can be rearranged as

$$\frac{(x-5)^2}{16} + \frac{(y+2)^2}{1} = 1.$$

This is the ellipse in standard position with $a = 4$, $b = 1$, translated to have centre $(5, -2)$, so a parametrisation is

$$x = 5 + 4\cos t, \quad y = -2 + \sin t$$
$$(0 \le t \le 2\pi).$$

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